Find The Volume V Of Liquid Needed To Fill A Sphere Of Radius R To Height H = R/3 .

Find The Volume V Of Liquid Needed To Fill A Sphere Of Radius R To Height H = R/3 . Determining the volume of liquid required to fill a specific portion of a sphere is a classic problem in calculus and geometry. When the sphere has a radius R and is filled up to a height H = R/3, the task is to compute the volume of the segment or cap of the sphere that corresponds to this height. This calculation is essential in various fields such as fluid mechanics, engineering, and physics, where understanding the volume of partially filled spherical containers or reservoirs is crucial. In this article, we will explore how to derive the volume V of the liquid needed, step by step, using integral calculus and geometric principles.

Understanding the Geometry of the Problem

Before delving into calculations, it's essential to visualize the problem and understand the geometric setup.

The Sphere and Its Cross-Sections

A sphere of radius R is a three-dimensional object with all points on its surface located at a distance R from the center. When filled to a height H = R/3, the liquid volume corresponds to a spherical cap or segment that extends from the top of the sphere down to a certain level.

The Height H and Its Significance

Given H = R/3, this height is measured from the topmost point of the sphere down to the liquid surface. Thus, the liquid occupies the upper portion of the sphere, forming a cap with height R/3.

Mathematical Approach to Finding the Volume

Calculating the volume involves setting up an integral that sums the infinitesimal slices of the spherical segment up to height H.

Coordinate System and Variables

Choose a coordinate system with the origin at the center of the sphere.
- The sphere's equation: $x^2 + y^2 + z^2 = R^2$.
- The vertical axis is along the z-axis, with the top point of the sphere at $z = R$.

Since the problem involves a horizontal cut at height $z = h$ (measured from the center), the volume of the cap above this plane can be found by integrating over the relevant region.

Expressing the Height H in Terms of the Coordinate System

The height of the cap from the top of the sphere is related to the z-coordinate:
- The plane that cuts the sphere at height H from the top corresponds to $z = R - H$.

Given $H = R/3$, the cutting plane is at $z = R - R/3 = 2R/3$.

Deriving the Volume of the Spherical Cap

The volume of a spherical cap of height h in a sphere of radius R can be obtained using the standard formula:

$$
V_{cap} = \frac{\pi h^2 (3R - h)}{3}
$$

This formula is derived from integral calculus by integrating cross-sectional areas of disks stacked along the height of the cap.

Applying the Formula for Our Case

In our problem, the height of the cap from the top is $h = R/3$.
- The volume of the cap (liquid) is:

$$
V = \frac{\pi (R/3)^2 (3R - R/3)}{3}
$$

Simplify numerator and denominator:

$$
V = \frac{\pi R^2 / 9 \times (3R - R/3)}{3}
$$

Calculate $3R - R/3$:

$$
3R - \frac{R}{3} = \frac{9R - R}{3} = \frac{8R}{3}
$$

Now, plug this back into the volume expression:

$$
V = \frac{\pi R^2 / 9 \times 8R/3}{3}
$$

Multiply numerator:

$$
V = \frac{\pi R^2 \times 8R}{9 \times 3} \times \frac{1}{3}
$$

\]

Simplify denominators:

\[
V = \frac{\pi \times 8 R^3}{27} \times \frac{1}{3} = \frac{\pi \times 8 R^3}{81}
\]

Finally,

\[
V = \frac{8 \pi R^3}{81}
\]

Therefore, the volume of liquid needed to fill the sphere up to height \(H = R/3 \) is:

\[
\boxed{
V = \frac{8 \pi R^3}{81}
}
\]

Alternative Method: Using Integration

While the above formula provides a quick way to compute the volume, understanding the integral approach offers deeper insight.

Setting Up the Integral

- Consider slices of the sphere at a height \(z \), where \(z \) ranges from the top of the sphere (\(z = R \)) down to the plane at \(z = 2R/3 \).
- The radius of each horizontal disk at height \(z \) is given by:

\[
r(z) = \sqrt{R^2 - z^2}
\]

- The differential volume element:

\[
dV = \pi r(z)^2 dz = \pi (R^2 - z^2) dz
\]

- The total volume of the cap:

\[
V = \int_{z=2R/3}^{z=R} \pi (R^2 - z^2) dz
\]

Performing the integral:

$$
V = \pi \int_{2R/3}^{R} (R^2 - z^2) dz
$$

Calculate the integral:

$$
V = \pi \left[R^2 z - \frac{z^3}{3} \right]_{z=2R/3}^{z=R}
$$

Plug in the limits:

$$
V = \pi \left(R^3 - \frac{R^3}{3} - \left(R^2 \times \frac{2R}{3} - \frac{\left(\frac{2R}{3} \right)^3}{3} \right) \right)
$$

Simplify step by step:

- $R^2 \times \frac{2R}{3} = \frac{2 R^3}{3}$

- $\left(\frac{2R}{3} \right)^3 = \frac{8 R^3}{27}$

- So,

$$
V = \pi \left(R^3 - \frac{R^3}{3} - \left(\frac{2 R^3}{3} - \frac{8 R^3}{81} \right) \right)
$$

Simplify the inner parentheses:

$$
\frac{2 R^3}{3} - \frac{8 R^3}{81} = \frac{54 R^3}{81} - \frac{8 R^3}{81} = \frac{46 R^3}{81}
$$

Now, the entire expression:

$$
V = \pi \left(R^3 - \frac{R^3}{3} - \frac{46 R^3}{81} \right)
$$

Express all terms with denominator 81:

$$
R^3 = \frac{81 R^3}{81}, \quad \frac{R^3}{3} = \frac{27 R^3}{81}
$$

Thus:

$$
V = \pi \left(\frac{81 R^3}{81} - \frac{27 R^3}{81} - \frac{46 R^3}{81} \right) = \pi \times \frac{81 R^3 - 27 R^3 - 46 R^3}{81}
$$

Simplify numerator:

$$
(81 - 27 - 46) R^3 = (8) R^3
$$

Finally:

$$
V = \pi \times \frac{8 R^3}{81} = \frac{8 \pi R^3}{81}
$$

This confirms our previous result obtained via the standard cap volume formula.

Practical Implications and Applications

Understanding how to calculate the volume of a spherical segment up to a given height has several practical applications:

- **Designing spherical tanks and reservoirs:** To determine how much liquid a spherical tank can hold up to a certain level.
- **Fluid mechanics:** Calculating the volume of liquid in a partially filled spherical container during operations or experiments.
- **Engineering applications:** Designing pressure vessels, balloons, or other spherical structures where volume calculations are essential for safety and efficiency.
- **Environmental science:** Estimating water volumes in natural or artificial spherical basins or ponds.

Summary

Calculating the volume of a liquid filling a sphere up to a certain height involves understanding the geometric properties of the sphere and applying integral calculus or known formulas for spherical caps. For

Frequently Asked Questions

How do I find the volume of liquid needed to fill a sphere of radius R up to a height H = R/3?

You need to compute the volume of the spherical cap formed up to height $H = R/3$, which is given by $V = (1/3)\pi h^2(3R - h)$. Substituting $h = R/3$, the volume becomes $V = (1/3)\pi(R/3)^2(3R - R/3)$.

What is the formula for the volume of a spherical cap with height h?

The volume of a spherical cap with height h in a sphere of radius R is $V = (1/3)\pi h^2(3R - h)$.

Given R and H = R/3, how do I compute the volume of liquid to fill up to that height?

Substitute $h = R/3$ into the spherical cap volume formula: $V = (1/3)\pi(R/3)^2(3R - R/3)$. Simplify to find the volume.

Can I use the spherical cap volume formula directly for H = R/3?

Yes. Plug in $h = R/3$ into the spherical cap volume formula to find the volume of liquid needed.

What is the simplified volume formula when filling a sphere up to height H = R/3?

Substituting $h = R/3$ into $V = (1/3)\pi h^2(3R - h)$ yields $V = (1/3)\pi(R/3)^2(3R - R/3)$. Simplify to get $V = (\pi R^3)/27$.

How does the volume change if I vary the height H in a sphere of radius R?

The volume of the filled region is given by the spherical cap formula $V = (1/3)\pi h^2(3R - h)$, which increases as H increases from 0 to R.

Is the volume formula for a spherical cap applicable for any height H less than R?

Yes. The formula $V = (1/3)\pi h^2(3R - h)$ applies for any h between 0 and R, representing the height of the cap.

What is the numerical value of the volume when R = 3 units

and H = R/3?

When R = 3, H = 1. Using $V = (1/3)\pi h^2(3R - h)$, with h=1: $V = (1/3)\pi(1)^2(33 - 1) = (1/3)\pi(1)(9 - 1) = (1/3)\pi(8) = (8/3)\pi$ units^3.

How do I derive the volume of a spherical cap from basic principles?

You integrate the cross-sectional areas of disks from the base up to height h, leading to the formula $V = (1/3)\pi h^2(3R - h)$.

Why is the formula for the spherical cap volume important in practical applications?

It helps in calculating liquid volumes in spherical containers, designing tanks, and understanding phenomena involving spherical shapes in engineering and physics.

Additional Resources

Find The Volume V Of Liquid Needed To Fill A Sphere Of Radius R To Height H = R/3

In the world of mathematics and engineering, understanding how to calculate the volume of a liquid contained within a specific portion of a sphere is a fundamental problem with practical applications ranging from designing tanks to modeling natural phenomena. Today, we delve into a classic geometric challenge: finding the volume V of liquid needed to fill a sphere of radius R up to a height H, where H equals R/3. This problem not only involves integrating the volume of a spherical segment but also offers insight into the interplay between geometry and calculus. Whether you're a student, engineer, or enthusiast, this comprehensive guide aims to clarify the process step-by-step, presenting the concepts in a reader-friendly yet technically precise manner.

Understanding the Geometric Setup

Before jumping into calculations, it's crucial to grasp the geometric configuration of the problem.

The Sphere and Its Dimensions

- Radius (R): The sphere has a radius R, which defines its size.
- Height (H): The liquid fills the sphere up to a height H from the bottom, with H given as \(H = \frac{R}{3} \).

Visually, imagine a perfect sphere sitting upright. The liquid fills from the bottom up to a certain horizontal plane, creating a spherical cap—a segment of the sphere cut off by a plane.

The Spherical Cap

A spherical cap is the portion of a sphere cut off by a plane parallel to its base. Its volume depends on the height of the cap (H) and the sphere's radius (R).

Key parameters:

- Height of the cap (H): The distance from the plane to the top of the sphere (or from the bottom to the cut plane).
- Base radius of the cap (a): The radius of the circular intersection between the plane and the sphere.
- Sphere radius (R): The original sphere's radius.

Given that the liquid fills up to height $H = R/3$, our goal is to compute the volume of the spherical cap corresponding to this height.

Mathematical Foundations: Volume of a Spherical Cap

The problem reduces to calculating the volume of a spherical cap with height H.

Formula for the Volume of a Spherical Cap

The volume V of a spherical cap of height H in a sphere of radius R is given by:

$$
V = \frac{\pi H^2 (3R - H)}{3}
$$

This formula elegantly relates the cap volume to the sphere's radius and the cap's height. It is derived using integral calculus, considering the sphere's geometry.

Deriving the Volume Step-by-Step

While the formula is well-known, understanding its derivation enhances comprehension and confidence in applying it to various problems.

Step 1: Coordinate System Setup

- Place the sphere centered at the origin (O) with coordinates (x, y, z).
- The sphere's equation: $x^2 + y^2 + z^2 = R^2$.

Assuming the height measurement is along the z-axis:

- The plane cutting the sphere to form the cap is at $z = h$, where h is from the bottom of the sphere.

Given that the total height of the liquid is H, and assuming the sphere's bottom is at $z = -R$, the plane intersects the sphere at $z = H - R$:

$$
h = H - R
$$

\]

Because the sphere extends from $z = -R$ to $z = R$, the cap's height from the bottom is:

$$
H = R + h
$$

In our problem, since the liquid fills up to $H = R/3$, then:

$$
h = H - R = \frac{R}{3} - R = - \frac{2R}{3}
$$

This indicates the cutting plane is below the sphere's center at $z = - \frac{2R}{3}$.

Step 2: Expressing Cap Volume via Integration

The volume of the cap is the integral of the cross-sectional disks from the bottom of the sphere up to the plane at $z = h$:

$$
V = \int_{z = -R}^{z = h} \pi [x^2 + y^2] \, dz
$$

From the sphere's equation:

$$
x^2 + y^2 = R^2 - z^2
$$

Thus:

$$
V = \pi \int_{z = -R}^{z = h} (R^2 - z^2) \, dz
$$

Calculating the integral:

$$
V = \pi \left[R^2 z - \frac{z^3}{3} \right]_{z = -R}^{z = h}
$$

Plugging in the limits:

$$
V = \pi \left(R^2 h - \frac{h^3}{3} \right) - \pi \left(R^2 (-R) - \frac{(-R)^3}{3} \right)
$$

Simplify the second term:

$$
R^2 (-R) = - R^3
$$

$$
(- R)^3 = - R^3
$$

So:

$$
V = \pi \left(R^2 h - \frac{h^3}{3} \right) - \pi \left(- R^3 - \frac{- R^3}{3} \right) = \pi \left(R^2 h - \frac{h^3}{3} + R^3 + \frac{R^3}{3} \right)
$$

Combine like terms:

$$
V = \pi \left(R^2 h + R^3 + \left(- \frac{h^3}{3} + \frac{R^3}{3} \right) \right)
$$

Alternatively, it's clearer to write the volume formula directly in terms of H:

$$
V = \frac{\pi H^2 (3 R - H)}{3}
$$

which is consistent with the earlier formula.

Applying the Formula to Our Specific H = R/3

Given $H = R/3$, substitute into the cap volume formula:

$$
V = \frac{\pi (R/3)^2 \left(3 R - R/3 \right)}{3}
$$

Simplify step-by-step:

1. Compute $(R/3)^2$:

$$
\left(\frac{R}{3} \right)^2 = \frac{R^2}{9}
$$

2. Simplify $3 R - R/3$:

$$
3 R - \frac{R}{3} = \frac{9 R}{3} - \frac{R}{3} = \frac{8 R}{3}
$$

3. Plug into the formula:

$$
V = \frac{\pi \times \frac{R^2}{9} \times \frac{8 R}{3}}{3}
$$

4. Multiply numerator:

$$
\pi \times \frac{R^2}{9} \times \frac{8 R}{3} = \pi \times \frac{8 R^3}{27}
$$

5. Divide by 3:

$$
V = \frac{\pi \times \frac{8 R^3}{27}}{3} = \pi \times \frac{8 R^3}{81}
$$

6. Final simplified volume:

$$
V = \frac{8 \pi R^3}{81}
$$

This is the volume of liquid required to fill the sphere up to height $H = R/3$.

Practical Implications and Applications

Understanding this calculation has broader implications:

- Engineering Design: When designing tanks or containers, calculating the volume of liquid for a given fill level helps determine capacity and material requirements.
- Natural Sciences: Modeling natural reservoirs or geological formations often involves calculating partial volumes of spherical or ellipsoidal regions.
- Manufacturing: Precise volume measurements are essential in industries like pharmaceuticals or food processing, where filling containers to specific levels is critical.

Moreover, the technique illustrated—deriving the volume of a spherical segment—is applicable to more complex geometries and can be extended to calculate volumes of irregular shapes with appropriate modifications.

Summary and Key Takeaways

- The problem involves calculating the volume of a spherical cap corresponding to a fill height $H = R/3$.
- The standard formula for the volume of a spherical cap is:

$$
V = \frac{\pi H^2 (3 R - H)}{3}
$$

- For $H = R/3$, the volume simplifies to:

$$
V = \frac{8 \pi R^3}{81}
$$

- This calculation combines geometric intuition with calculus, illustrating how integral methods underpin many practical volume computations.

Conclusion

Calculating the volume of liquid needed to fill a sphere up to a certain height exemplifies the harmony between geometry and calculus. By understanding the properties of spherical caps and leveraging well-established formulas, one can swiftly determine the amount of liquid required for various fill levels. This knowledge is invaluable across scientific disciplines and engineering fields, enabling precise design, measurement, and analysis. As we've seen, even a seemingly simple question about filling

Find The Volume V Of Liquid Needed To Fill A Sphere Of Radius R To Height H R 3

Find The Volume V Of Liquid Needed To Fill A Sphere Of Radius R To Height H R 3

Related Articles

- Graph The Line Y = 5x 1, Then Name The Slope And Y-intercept By Looking At The Graph.
- Make A Slide About Pearl Harbor Up To 5 Slides Each One About Something Diferrent
- The Major Distinction Between Gay Or Lesbian Families And Heterosexual Families Is?

Back to Home