In Problems 21 And 22 Solve The Given Initial-value Problem. 21.x2dxdy2xy=3y4,y(1)=21

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When delving into the realm of differential equations, especially initial-value problems, understanding the methods to solve them is essential for students and professionals alike. The problem presented — solving the differential equation \(x^2 \frac{dy}{dx} = 2xy + 3y^4\) with the initial condition \(y(1) = 21\) — is a classic example that illustrates the application of various techniques in differential equations. This article provides a comprehensive guide to solving this initial-value problem step by step, exploring the underlying concepts, methods, and solutions while optimizing for clarity and SEO relevance.

Understanding the Initial-Value Problem (IVP)

Before diving into the solution, it's crucial to understand what an initial-value problem entails and why it's significant in differential equations.

What is an Initial-Value Problem?

An initial-value problem involves finding a function \(y(x) \) that satisfies a differential equation along with an initial condition specifying the value of \(y \) at a particular point \(x = x_0 \). Formally, it can be written as:

\[
\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0
\]

In our specific problem:

\[
x^2 \frac{dy}{dx} = 2xy + 3y^4, \quad y(1) = 21
\]

The goal is to find the function \(y(x) \) that satisfies this differential equation and passes through the point \((1, 21) \).

Significance of Initial Conditions

Initial conditions are vital because, in many differential equations, there are infinitely many solutions. The initial condition helps determine the unique solution relevant to the problem's context.

Step-by-Step Solution to the Differential Equation

The key to solving the given initial-value problem lies in recognizing the structure of the differential equation and applying appropriate methods such as substitution and separation of variables.

1. Rewrite the Differential Equation

Given:

$$
x^2 \frac{dy}{dx} = 2xy + 3y^4
$$

Divide both sides by x^2 to isolate $\frac{dy}{dx}$:

$$
\frac{dy}{dx} = \frac{2xy + 3y^4}{x^2}
$$

Expressed as:

$$
\frac{dy}{dx} = \frac{2xy}{x^2} + \frac{3y^4}{x^2} = \frac{2y}{x} + \frac{3y^4}{x^2}
$$

This form suggests the equation is nonlinear but can be approached through substitution.

2. Recognize the Type of Differential Equation

The differential equation:

$$
\frac{dy}{dx} = \frac{2y}{x} + \frac{3y^4}{x^2}
$$

resembles a Bernoulli differential equation because it involves terms like y and y^4.

A Bernoulli equation has the form:

$$
\frac{dy}{dx} + P(x) y = Q(x) y^n
$$

Rearranged:

$$
\frac{dy}{dx} - \left(-\frac{2}{x}\right) y = \frac{3}{x^2} y^4
$$

which can be written as:

$$
\frac{dy}{dx} + P(x) y = Q(x) y^n
$$

with:

$$
P(x) = -\frac{2}{x}, \quad Q(x) = \frac{3}{x^2}, \quad n = 4
$$

Since $n \neq 0, 1$, Bernoulli's method applies.

3. Transform the Bernoulli Equation

Set:

$$
z = y^{1 - n} = y^{-3}
$$

Then:

$$
\frac{dz}{dx} = -(n - 1) y^{-n} \frac{dy}{dx} = -3 y^{-4} \frac{dy}{dx}
$$

Rearranged:

$$
\frac{dy}{dx} = -\frac{1}{3} y^{4} \frac{dz}{dx}
$$

Substitute into the original differential equation:

$$
-\frac{1}{3} y^4 \frac{dz}{dx} + P(x) y = Q(x) y^n
$$

But since $y^{1 - n} = z \Rightarrow y = z^{-\frac{1}{3}}$, we need to express everything in terms of z:

$$
\frac{dy}{dx} = -\frac{1}{3} y^4 \frac{dz}{dx}
$$

Plug into the Bernoulli form:

$$
-\frac{1}{3} y^4 \frac{dz}{dx} + P(x) y = Q(x) y^n
$$

Expressed in z:

$$
-\frac{1}{3} y^4 \frac{dz}{dx} + P(x) y = Q(x) y^4
$$

Divide through by y^4:

$$
-\frac{1}{3} \frac{dz}{dx} + P(x) y^{-3} = Q(x)
$$

Recall $y^{-3} = z$:

$$
-\frac{1}{3} \frac{dz}{dx} + P(x) z = Q(x)
$$

Now, substitute $P(x) = -\frac{2}{x}$, $Q(x) = \frac{3}{x^2}$:

$$
-\frac{1}{3} \frac{dz}{dx} - \frac{2}{x} z = \frac{3}{x^2}
$$

Multiply both sides by -3 to simplify:

$$
\frac{dz}{dx} + \frac{6}{x} z = -\frac{9}{x^2}
$$

This is a linear first-order differential equation in z.

4. Solve the Linear Differential Equation for z

The standard form:

$$
\frac{dz}{dx} + P(x) z = Q(x)
$$

where:

\[
P(x) = \frac{6}{x}, \quad Q(x) = -\frac{9}{x^2}
\]

Calculate the integrating factor \(\mu(x) \):

\[
\mu(x) = e^{\int P(x) dx} = e^{\int \frac{6}{x} dx} = e^{6 \ln |x|} = |x|^{6}
\]

Assuming \(x > 0 \), simplifies to:

\[
\mu(x) = x^6
\]

Multiply the entire differential equation by \(\mu(x) \):

\[
x^6 \frac{dz}{dx} + x^6 \frac{6}{x} z = -9 x^4
\]

Simplify:

\[
x^6 \frac{dz}{dx} + 6 x^5 z = -9 x^4
\]

Observe that the left side is the derivative of \(x^6 z \):

\[
\frac{d}{dx} (x^6 z) = -9 x^4
\]

Integrate both sides:

\[
x^6 z = \int -9 x^4 dx + C
\]

Calculate the integral:

\[
\int -9 x^4 dx = -9 \cdot \frac{x^5}{5} = -\frac{9}{5} x^5
\]

So,

\[

x^6 z = -\frac{9}{5} x^5 + C
\]

Solve for \(z \):

\[
z = -\frac{9}{5} \frac{x^5}{x^6} + \frac{C}{x^6} = -\frac{9}{5} \frac{1}{x} + \frac{C}{x^6}
\]

Recall that \(z = y^{-3} \), so:

\[
y^{-3} = -\frac{9}{5} \frac{1}{x} + \frac{C}{x^6}
\]

Express \(y \):

\[
y = \left(-\frac{9}{5} \frac{1}{x} + \frac{C}{x^6} \right)^{-\frac{1}{3}}
\]

Applying the Initial Condition

To find the constant \(C \), use the initial condition \(y(1) = 21 \):

\[
21 = \left(-\frac{9}{5} \cdot 1 + C \cdot 1 \right)^{-\frac{1}{3}}
\]

Simplify inside the parentheses:

\[
21 = \left(-\frac{

Frequently Asked Questions

What is the initial-value problem given in Problem 21?

The initial-value problem is to solve the differential equation x^2 dx dy 2xy = 3 y^4 with the initial condition y(1) = 21.

How do you interpret the differential equation in Problem 21?

It appears to be a typo or formatting error; the intended differential equation is likely x^2 dy/dx + 2xy

= 3 y^4, which is a first-order differential equation to solve with the initial condition y(1) = 21.

What method can be used to solve the initial-value problem in Problem 21?

Since the differential equation appears to be nonlinear, a substitution such as v = y^{-3} or methods for Bernoulli equations can be used to simplify and solve it.

What is the first step in solving the problem's differential equation?

Rearranging the equation into a standard form, identifying if it's Bernoulli or linear, and then applying appropriate substitution or integrating factor methods.

How do you apply the initial condition y(1) = 21 in solving the problem?

Once the general solution is obtained, substitute x = 1 and y = 21 to find the integration constant, ensuring the particular solution satisfies the initial condition.

Is the differential equation in Problem 21 separable or exact?

Based on the form, it appears nonlinear and likely not separable or exact; it may require substitution methods such as Bernoulli's equation approach.

What are the key challenges in solving the initial-value problem in Problem 21?

Handling the nonlinear term y^4 and correctly applying substitution methods to reduce the differential equation to solvable form while satisfying the initial condition.

What is the significance of solving Problems 21 and 22 in the context of differential equations?

These problems help practice solving initial-value problems involving nonlinear differential equations, reinforcing methods like substitution, integration, and applying initial conditions accurately.

Additional Resources

In the realm of differential equations, initial-value problems (IVPs) serve as fundamental tools for modeling a wide array of physical phenomena, from physics and engineering to biology and economics. These problems, which specify the behavior of a function and its derivatives at a given point, often pose intriguing mathematical challenges that require methodical approaches and nuanced understanding. In this article, we delve into the detailed solution of two such initial-value problems, specifically Problems 21 and 22, with a focus on problem 21: solving the differential equation \(x^2 \, dx \, dy^2 + 2xy = 3 y^4

\) with the initial condition $y(1) = 21$. We will analyze the formulation, solution techniques, and the implications of these problems in a comprehensive, structured manner, aiming to clarify the intricacies involved and the significance of their solutions.

Understanding the Initial-Value Problem (IVP) in Context

The Significance of IVPs in Differential Equations

Initial-value problems are a class of differential equations accompanied by specified initial conditions, typically values of the unknown function and possibly its derivatives at a particular point. They are crucial because:

- They enable the determination of a unique solution fitting the physical, biological, or economic situation modeled.
- They encapsulate real-world constraints and initial states, making the solutions practically relevant.
- They serve as foundational problems for numerical methods and analytical techniques.

Overview of the Problem at Hand

In Problem 21, the differential equation:

$$x^2 \, dx \, dy^2 + 2xy = 3 y^4$$

is coupled with the initial condition:

$$y(1) = 21$$

This problem involves an unconventional differential expression, combining differentials dx and dy and involving the square of the differential dy^2, which suggests a need for careful interpretation and manipulation to convert it into a solvable form.

Reformulating the Differential Equation

Interpreting the Differential Expression

The notation:

$$x^2 \, dx \, dy^2 + 2xy = 3 y^4$$

may initially seem confusing because of the term dy^2. Typically, in differential equations, dy

represents an infinitesimal change in \(y \) with respect to \(x \), and \(dy^2 \) could be interpreted as \((dy)^2 \), the square of the differential.

Assuming the standard notation:

- \(dy \) is an differential of \(y \),
- \(dy^2 \) represents \((dy)^2 \),
- \(dx \) is the differential of \(x \).

Thus, the term \(dx \, dy^2 \) can be interpreted as \(dx \times (dy)^2 \). To make sense of this, the equation can be viewed as an expression involving differentials:

\[x^2 \, dx \, (dy)^2 + 2xy = 3 y^4 \]

which hints at a relation involving the differentials in a combined manner.

Key insight: To interpret this correctly, it may be advantageous to consider the total differential or to divide through by \(dx \) to convert the equation into an ordinary differential equation (ODE) involving \(y \) and \(dy/dx \).

Transforming into an Ordinary Differential Equation

Suppose \(y \) is a function of \(x \), \(y = y(x) \). Then, \(dy = \frac{dy}{dx} dx \). This allows us to rewrite the differentials:

\[(dy)^2 = \left(\frac{dy}{dx} \right)^2 dx^2 \]

Given that, the original expression becomes:

\[x^2 \, dx \times \left(\frac{dy}{dx} \right)^2 dx^2 + 2xy = 3 y^4 \]

But this seems inconsistent unless we interpret the original as an expression involving differentials in a differential form:

\[x^2 \, dx \, (dy)^2 + 2xy = 3 y^4 \]

Alternatively, perhaps the problem intends for us to interpret the differential as a form, possibly implying a substitution or a specific method such as differential forms or an integrating factor.

Alternatively, considering the pattern of similar differential equations, perhaps the original expression intends the differential form:

\[x^2 y'' + 2xy = 3 y^4 \]

which appears more manageable and standard in differential equations.

Clarifying the Form of the Differential Equation

Likely Correction and Standard Form

Given the form and the initial condition, the most logical assumption is that the original problem is a second-order differential equation:

$$ x^2 y'' + 2xy = 3 y^4 $$

with initial condition:

$$ y(1) = 21 $$

This form aligns with standard differential equations encountered in textbooks and allows for established solution methods.

Reasoning:

- The presence of y'' (second derivative) aligns with the notation dy^2 as a misinterpretation.
- The initial condition $y(1) = 21$ applies at a specific point, consistent with a second-order ODE problem.

Thus, we proceed under the assumption that the differential equation is:

$$ x^2 y'' + 2xy = 3 y^4 $$

Solving the Differential Equation

Methodology Overview

The differential equation:

$$ x^2 y'' + 2xy = 3 y^4 $$

is nonlinear, but it exhibits a structure similar to an Euler-Cauchy (equidimensional) equation, which suggests substitution techniques or reduction methods.

Step 1: Recognize the form
- The coefficients x^2 and $2x$ suggest an Euler-Cauchy type.
- The nonlinear term $3 y^4$ complicates direct solutions, but substitution may linearize part of the problem.

Step 2: Substitution to reduce order
Let $p = y'$. Then, $y'' = p' = \frac{dp}{dx}$.

Rewriting the differential equation:

$$ x^2 p' + 2x p = 3 y^4 $$

Now, the challenge is to relate y and p. Since $p = y'$, maybe a substitution involving $z = y$ or $v = y^{n}$ could be advantageous.

Employing a Substitution: $v = y^{m}$

Given the nonlinear nature, a substitution such as $v = y^{k}$ might simplify the nonlinear term y^4.

Suppose:

$$ v = y^{m} $$
then,

$$ y = v^{1/m} $$

and

$$ y' = \frac{1}{m} v^{(1/m) - 1} v' $$

Similarly, derivatives involve v', and the goal is to choose m to simplify y^4.

Choosing $m = 4$:

$$ v = y^4 $$

then,

$$ y = v^{1/4} $$

and

$$ y' = \frac{1}{4} v^{-3/4} v' $$

and

$$ y'' = \frac{d}{dx} y' $$

which involves derivatives of v.

However, this approach might complicate the derivatives further, so perhaps a better route is to look for similarity solutions or to attempt reduction via substitution $t = \ln x$.

Transformation to a Simpler Equation: $t = \ln x$

Set:

$$ t = \ln x \Rightarrow x = e^t $$

then, derivatives transform as:

$$ \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} $$

and

$$ y' = \frac{1}{x} y_t $$

Similarly,

$$ y'' = \frac{d}{dx} y' = \frac{d}{dx} \left(\frac{1}{x} y_t \right) $$

which simplifies to:

$$ y'' = - \frac{1}{x^2} y_t + \frac{1}{x^2} y_{tt} $$

Substituting into the original differential equation:

$$ x^2 y'' + 2xy = 3 y^4 $$

becomes:

$$ x^2 \left(- \frac{1}{x^2} y_t + \frac{1}{x^2} y_{tt} \right) + 2x y = 3 y^4 $$

which simplifies to:

$$ - y_t + y_{tt} + 2x y = 3 y^4 $$

Expressed in t:

$$ y_{tt} - y_t + 2 e^{t} y = 3 y^4 $$

This is a nonlinear second-order ODE in $y(t)$. The presence of e^{t} complic

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