Is The Sequence An= (4)" A Solution Of The Recurrence Relation An = 8an-1 - 16an-2

Is The Sequence An= (4)^n A Solution Of The Recurrence Relation An = 8an-1 - 16an-2

Understanding whether a specific sequence satisfies a given recurrence relation is a fundamental aspect of discrete mathematics and sequence analysis. In this article, we will explore the sequence An = (4)^n and determine if it is a solution to the recurrence relation An = 8an-1 - 16an-2. We will delve into the theory behind recurrence relations, the method of solving them, and then apply these concepts step-by-step to verify whether the sequence meets the relation's criteria.

Introduction to Recurrence Relations

What is a Recurrence Relation?

A recurrence relation is an equation that defines each term of a sequence in terms of previous terms. It provides a way to generate the sequence iteratively. For example, a simple recurrence relation might be:

An = c1 An-1 + c2 An-2 + ... + ck An-k + d

where c1, c2, ..., ck, and d are constants, and initial conditions are required to compute the sequence uniquely.

Importance of Solving Recurrence Relations

Solving recurrence relations allows us to find a closed-form expression for the sequence, which can be easier to analyze, compute, and interpret. It also helps in understanding the long-term behavior of sequences, such as whether they grow exponentially, oscillate, or stabilize.

Given Recurrence Relation and Sequence

Our focus is on the recurrence relation:

An = 8an-1 - 16an-2

and the sequence:

$A_n = (4)^n$

The question is: Does the sequence $A_n = (4)^n$ satisfy this recurrence relation?

To answer this, we will substitute the sequence into the recurrence relation and verify if the equation holds for all n, provided the initial conditions are compatible.

Methodology for Verification

The verification involves the following steps:

1. Express A_n, A_{n-1}, and A_{n-2} in terms of $(4)^n$.
2. Substitute these into the recurrence relation.
3. Simplify and check if the equality holds for all n.

Step 1: Express the Terms

Given:

$A_n = (4)^n$

then,

$A_{n-1} = (4)^{n-1}$

and

$A_{n-2} = (4)^{n-2}$

Step 2: Substitute into the Recurrence Relation

The recurrence relation is:

$A_n = 8A_{n-1} - 16A_{n-2}$

Plugging in the expressions:

$(4)^n = 8 (4)^{n-1} - 16 (4)^{n-2}$

Step 3: Simplify the Equation

Let's simplify the right side:

- $8 (4)^{n-1} = 8 (4^{n-1}) = 8 (4^{n-1})$

- $16 (4)^{n-2} = 16 (4^{n-2})$

Recall that:

$4^{n-1} = 4^{n} / 4 = (4^{n}) / 4$

Similarly,

$4^{n-2} = 4^{n} / 4^2 = (4^{n}) / 16$

Substitute these back:

Right side = $8 (4^{n} / 4) - 16 (4^{n} / 16)$

Simplify each term:

- $8 (4^{n} / 4) = (8 / 4) 4^{n} = 2 4^{n}$

- $16 (4^{n} / 16) = (16 / 16) 4^{n} = 1 4^{n}$

Therefore, the right side becomes:

$2 4^{n} - 1 4^{n} = (2 - 1) 4^{n} = 1 4^{n}$

Now, compare with the left side:

Left side = 4^{n}

Right side = 4^{n}

They are equal for all n.

Conclusion of verification:

Since the left side equals the right side for all integer n, the sequence An = $(4)^n$ satisfies the recurrence relation An = 8an-1 - 16an-2, provided initial conditions are consistent.

Initial Conditions and Consistency

To fully validate the sequence as a solution, initial conditions must be compatible. For the sequence An = $(4)^n$, we have:

- For n=0:

A0 = $(4)^0 = 1$

- For $n=1$:

$A1 = (4)^1 = 4$

Now, check if these initial conditions satisfy the recurrence relation:

For $n=2$:

$An = 8An-1 - 16An-2$

Substitute $n=2$:

$A2 = 8 A1 - 16 A0$

Calculate:

$A2 = 8 4 - 16 1 = 32 - 16 = 16$

But from the sequence:

$A2 = (4)^2 = 16$

which matches perfectly.

Similarly, for $n=3$:

$A3 = 8 A2 - 16 A1$

Calculate:

$A3 = 8 16 - 16 4 = 128 - 64 = 64$

From the sequence:

$A3 = (4)^3 = 64$

Again, it matches.

This consistency holds for all n, confirming that the sequence $An = (4)^n$ is a particular solution satisfying the initial conditions.

General Solution to the Recurrence Relation

The recurrence relation:

$An = 8An-1 - 16An-2$

is a linear homogeneous recurrence relation with constant coefficients. Its characteristic equation is:

$r^2 - 8r + 16 = 0$

Solve for r:

$r^2 - 8r + 16 = 0$

Discriminant:

$D = (-8)^2 - 4 \cdot 1 \cdot 16 = 64 - 64 = 0$

Since D=0, the roots are real and repeated:

$r = 8 / 2 = 4$

Thus, the general solution of the recurrence relation is:

$A_n = (C_1 + C_2 n) 4^n$

where C_1 and C_2 are constants determined by initial conditions.

In our case, since $A_n = (4)^n$ is a solution, it corresponds to the particular solution with $C_1=1$ and $C_2=0$.

Implications and Applications

Understanding whether a sequence solves a recurrence relation has practical implications:

- Predictive Modeling: Many real-world phenomena modeled by recurrence relations include population growth, financial calculations, and computer algorithms. Recognizing solutions helps in predicting future terms efficiently.

- Algorithm Optimization: In computer science, recurrence relations often describe recursive algorithms. Knowing explicit solutions allows optimization and analysis of algorithm complexity.

- Mathematical Analysis: Identifying solutions aids in understanding the behavior of sequences, such as exponential growth, decay, oscillations, or convergence.

Summary and Key Takeaways

- The sequence $A_n = (4)^n$ satisfies the recurrence relation $A_n = 8a_{n-1} - 16a_{n-2}$, as verified by substitution and simplification.

- The characteristic equation associated with the recurrence has a repeated root at r=4, leading to the general solution involving terms of the form $(C_1 + C_2 n) 4^n$.

- Initial conditions consistent with the sequence confirm that An = (4)^n is a particular solution.

- Recognizing solutions to recurrence relations is essential in various scientific and engineering contexts.

Conclusion

In conclusion, the sequence An = (4)^n is indeed a solution to the recurrence relation An = 8an-1 - 16an-2. Its form aligns perfectly with the characteristic roots of the relation, and the initial conditions confirm its validity. This example illustrates the process of verifying solutions to recurrence relations, emphasizing the importance of substitution, algebraic simplification, and understanding characteristic equations. Mastery of these techniques is invaluable for students and professionals working in mathematics, computer science, and related fields, enabling them to analyze and predict the behavior of complex sequences efficiently.

If you are dealing with similar recurrence relations, remember to:

- Derive and solve the characteristic equation.
- Find the general solution based on roots.
- Verify potential solutions through substitution.
- Use initial conditions to determine specific solution constants.

By following these steps, you can confidently identify whether a sequence satisfies a given recurrence relation and understand its underlying structure.

Frequently Asked Questions

Is the sequence An = 4 a solution to the recurrence relation An = 8An-1 - 16An-2?

No, the sequence An = 4 is not a solution because substituting An = 4 into the recurrence results in a contradiction unless initial conditions are specifically chosen. Generally, a constant sequence An = c is a solution only if c satisfies the relation, which it does not in this case.

What is the characteristic equation associated with the recurrence relation An = 8An-1 - 16An-2?

The characteristic equation is $r^2 - 8r + 16 = 0$.

What are the roots of the characteristic equation for

the recurrence relation?

The roots are r = 4 and r = 4, which is a repeated root.

What is the general solution to the recurrence relation An = 8An-1 - 16An-2?

Since there is a repeated root r = 4, the general solution is An = (A 4^n) + (B n 4^n), where A and B are constants determined by initial conditions.

Can the constant sequence An = 4 be a particular solution to the recurrence relation?

Yes, if initial conditions are chosen such that A and B satisfy the particular solution An = 4, then it can be a solution, but generally, the constant sequence does not satisfy the recurrence unless the recurrence is specifically designed for that.

How do initial conditions affect whether An = 4 is a solution?

Initial conditions determine the constants A and B in the general solution. If they are set appropriately, the sequence can be constant and equal to 4; otherwise, it won't satisfy the recurrence for all n.

What is the particular solution when An = 4 is substituted into the recurrence relation?

Substituting An = 4 into the recurrence gives 4 = 8 4 - 16 4, which simplifies to 4 = 32 - 64 = -32, so it does not satisfy the recurrence unless initial conditions are adjusted.

Is the sequence An = 4 a homogeneous or particular solution?

An = 4 is neither a homogeneous nor a particular solution in the general case; it can be a particular solution only under specific initial conditions.

What method can be used to find the explicit formula for An in this recurrence?

The characteristic equation method is used, solving for roots and then constructing the general solution based on the nature of the roots.

Additional Resources

Is the Sequence An = 4^n a Solution to the Recurrence Relation An = 8An-1 - 16An-2?

Understanding whether a particular sequence satisfies a given recurrence relation is a fundamental task in discrete mathematics and mathematical analysis. It often involves analyzing the sequence's explicit formula, deriving the associated characteristic equation, and verifying that the sequence's terms align with the recurrence's constraints. In this article, we delve into the question: Is the sequence An = 4^n a solution of the recurrence relation An = 8An-1 - 16An-2? We will explore the theoretical underpinnings, perform detailed calculations, and interpret the results to provide a comprehensive understanding.

Introduction to Recurrence Relations and Sequences

What Are Recurrence Relations?

Recurrence relations are equations that define sequences recursively, meaning each term is expressed as a function of one or more previous terms. They are fundamental in modeling various phenomena, including population dynamics, algorithms, and financial calculations. A typical linear homogeneous recurrence relation with constant coefficients has the form:

$$ An = c_1 An-1 + c_2 An-2 + \dots + c_k An-k $$

where $c_1, c_2, ..., c_k$ are constants, and initial conditions specify the first few terms.

Explicit Form of Sequences

While recurrence relations define sequences recursively, explicit formulas directly compute the nth term without reference to prior terms. Deriving explicit formulas involves solving the characteristic equation associated with the recurrence relation.

The Given Sequence and Recurrence Relation

Sequence: $An = 4^n$

The sequence under consideration is:

$$ An = 4^n $$

which is a geometric sequence with common ratio 4, starting at \(n=0 \) (assuming standard indexing).

Recurrence Relation: \(An = 8 An-1 - 16 An-2 \)

The recurrence relation given is:

\[An = 8 An-1 - 16 An-2 \]

This is a second-order linear homogeneous recurrence relation with constant coefficients.

Methodology for Verification

To determine whether \(An = 4^n \) satisfies the recurrence relation, the standard approach involves:

- Substituting the explicit form \(An = 4^n \) into the recurrence relation.
- Checking if the relation holds for all \(n \) (or at least some representative values, then generalizing).
- Analyzing the characteristic equation associated with the recurrence to see if \(4^n \) corresponds to one of its solution components.

Step-by-Step Analysis

Substituting \(An = 4^n \) into the recurrence relation

Let's perform this substitution:

\[An = 4^n \]
\[An-1 = 4^{n-1} \]
\[An-2 = 4^{n-2} \]

Plugging into the recurrence:

\[4^n \stackrel{?}{=} 8 \times 4^{n-1} - 16 \times 4^{n-2} \]

We can simplify the right side:

\[8 \times 4^{n-1} = 8 \times 4^{n-1} \]

$$ 16 \times 4^{n-2} = 16 \times 4^{n-2} $$

Expressing both terms with the same base:

$$ 4^{n-1} = 4^{n} / 4 $$
$$ 4^{n-2} = 4^{n} / 4^2 = 4^{n} / 16 $$

Rewriting the right side:

$$ 8 \times 4^{n-1} = 8 \times \frac{4^{n}}{4} = 8 \times 4^{n} \times \frac{1}{4} = 2 \times 4^{n} $$

Similarly,

$$ 16 \times 4^{n-2} = 16 \times \frac{4^{n}}{16} = 4^{n} $$

Now, the recurrence relation becomes:

$$ 4^{n} \stackrel{?}{=} 2 \times 4^{n} - 4^{n} $$

Simplify the right side:

$$ 2 \times 4^{n} - 4^{n} = (2 - 1) \times 4^{n} = 1 \times 4^{n} = 4^{n} $$

This matches the left side exactly:

$$ 4^{n} = 4^{n} $$

Conclusion: The sequence $An = 4^n$ satisfies the recurrence relation for all n.

Understanding the Implications

Verification of the Solution

The direct substitution confirms that $An = 4^n$ is indeed a solution to the recurrence relation:

$$ An = 8 An-1 - 16 An-2 $$

This means that the explicit sequence 4^n belongs to the general solution space of the recurrence.

Connection to the Characteristic Equation

The next step is to understand this solution in the context of the characteristic equation, which provides a systematic way to identify all solutions.

The recurrence:

$$ An = 8 An-1 - 16 An-2 $$

has the characteristic polynomial:

$$ r^2 - 8 r + 16 = 0 $$

Solving:

$$ r^2 - 8 r + 16 = 0 $$
$$ (r - 4)^2 = 0 $$

which has a repeated root:

$$ r = 4 $$

Implication: The general solution to the recurrence is:

$$ An = (A) \times 4^{n} + (B \times n) \times 4^{n} $$

where A and B are constants determined by initial conditions.

Since 4^n appears as a fundamental solution associated with the root $r=4$, the sequence $An=4^n$ is a particular solution corresponding to the dominant root.

General Solution and Particular Solutions

Form of the General Solution

Given the characteristic root $r=4$ with multiplicity 2, the general solution is:

$$ An = \left(C_1 + C_2 n \right) 4^{n} $$

for arbitrary constants C_1, C_2.

Matching the Sequence $An=4^n$

The sequence $An=4^n$ corresponds to setting:

$$C_1 = 1$$
$$C_2 = 0$$

which is a specific solution within the general family.

Additional Considerations and Applications

Initial Conditions and Uniqueness

While we've confirmed that $An=4^n$ satisfies the recurrence relation, the initial conditions determine its status as a particular solution. For example, suppose the initial conditions are:

$$A_0 = 1, \quad A_1 = 4$$

then:

$$A_0 = 4^0 = 1$$
$$A_1 = 4^1 = 4$$

which are consistent with the sequence 4^n.

Alternatively, different initial conditions would lead to different constants C_1, C_2, but the sequence 4^n remains a valid solution.

Broader Implications in Mathematical Modeling

Sequences like 4^n often emerge in contexts with exponential growth or decay, such as population models, radioactive decay, and financial calculations. Recognizing that such sequences satisfy specific linear recurrence relations allows for leveraging algebraic techniques to analyze their behavior, derive closed-form expressions, and predict long-term trends.

Conclusion: Final Assessment

Through direct substitution and characteristic equation analysis, it is clear that the sequence \(An=4^n \) is indeed a solution to the recurrence relation:

\[An = 8 An-1 - 16 An-2 \]

This conclusion underscores the importance of understanding the relationship between explicit formulas and their recursive definitions. Recognizing that geometric sequences often correspond to roots of the characteristic polynomial simplifies the process of solving recurrence relations and analyzing their solutions comprehensively.

In summary:

- Explicit verification confirms the sequence satisfies the recurrence for all \(n \).
- Characteristic equation analysis reveals the sequence's role as a fundamental solution associated with a repeated root.
- Implications extend to the general solution form and potential applications in modeling real-world phenomena.

Ultimately, the sequence \(An=4^n \) exemplifies how explicit formulas and recurrence relations intertwine, highlighting the elegance and power of algebraic methods in discrete mathematics.

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