Let V1= 0 0 4 , V2= 0 4 12 , And V3= 6 5 8 . Does V1,v2,v3 Span 3? Why Or Why Not?

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Introduction

Understanding whether a set of vectors spans a particular space is a fundamental concept in linear algebra. In this context, the question posed is whether the vectors V1 = (0, 0, 4), V2 = (0, 4, 12), and V3 = (6, 5, 8) span the three-dimensional space $\mathbb{R}^3$. To answer this, we need to analyze the linear independence of these vectors and determine if they can generate any arbitrary vector in $\mathbb{R}^3$.

This article explores the concept of span, linear independence, and how to determine whether these vectors form a basis for $\mathbb{R}^3$. We will walk through the process step-by-step, including setting up and solving the relevant equations, and conclude whether V1, V2, and V3 span $\mathbb{R}^3$ or not.

Understanding the Concepts: Span, Linear Independence, and Basis

What is the Span of a Set of Vectors?

The span of a set of vectors is the set of all possible linear combinations of those vectors. For a set of vectors V1, V2, ..., Vn in $\mathbb{R}^3$, their span is the collection of all vectors that can be written in the form:

$$c_1 V_1 + c_2 V_2 + \cdots + c_n V_n$$

where $c_1, c_2, ..., c_n$ are scalars.

If the span of a set of vectors equals $\mathbb{R}^3$, then the vectors are said to span the entire three-dimensional space.

Linear Independence and Dependence

A set of vectors is linearly independent if no vector can be written as a linear combination of the others. Conversely, if at least one vector can be expressed as such, the vectors are linearly dependent.

In the context of spanning $\mathbb{R}^3$:

- If three vectors are linearly independent, they form a basis for $\mathbb{R}^3$ and thus span the space.
- If they are linearly dependent, they do not span all of $\mathbb{R}^3$.

Analyzing the Given Vectors

Given vectors:

- V1 = (0, 0, 4)
- V2 = (0, 4, 12)
- V3 = (6, 5, 8)

Our goal is to determine whether these vectors span $\mathbb{R}^3$, i.e., whether any arbitrary vector (x, y, z) in $\mathbb{R}^3$ can be expressed as a linear combination of V1, V2, and V3.

Setting Up the Mathematical Framework

To check whether V1, V2, and V3 span $\mathbb{R}^3$, we consider the matrix formed by placing these vectors as columns:

$$
A = \begin{bmatrix}
0 & 0 & 6 \\
0 & 4 & 5 \\
4 & 12 & 8
\end{bmatrix}
$$

The question reduces to analyzing the rank of this matrix. If the rank of matrix A is 3, the vectors are linearly independent and span $\mathbb{R}^3$. If the rank is less than 3, they do not.

Alternatively, we can check whether the linear system:

$$
c_1 V_1 + c_2 V_2 + c_3 V_3 = (x, y, z)
$$

has solutions for all (x, y, z).

Calculating the Determinant and Rank

Step 1: Form the Matrix

$$
A = \begin{bmatrix}
0 & 0 & 6 \\
0 & 4 & 5 \\
4 & 12 & 8
\end{bmatrix}
$$

Note: The vectors are columns, so the matrix is:

$$
A = \begin{bmatrix}
V_{1x} & V_{2x} & V_{3x} \\
V_{1y} & V_{2y} & V_{3y} \\
V_{1z} & V_{2z} & V_{3z}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 6 \\
0 & 4 & 5 \\
4 & 12 & 8
\end{bmatrix}
$$

Alternatively, sometimes vectors are taken as rows, but for the purpose of linear independence, the choice of columns or rows doesn't affect the rank.

Step 2: Calculate the Determinant of the Matrix

Calculating the determinant of A:

$$
\det(A) = \begin{vmatrix}
0 & 0 & 6 \\
0 & 4 & 5 \\
4 & 12 & 8
\end{vmatrix}
$$

Using cofactor expansion along the first row:

$$
\det(A) = 0 \times C_{11} - 0 \times C_{12} + 6 \times C_{13}
$$

Where C_{13} is the cofactor for element in first row, third column:

$$
C_{13} = (-1)^{1+3} \times \det \begin{bmatrix}
0 & 4 \\
4 & 12
\end{bmatrix}
= (+1) \times (0 \times 12 - 4 \times 4) = (0 - 16) = -16
$$

Thus,

$$
\det(A) = 6 \times (-16) = -96
$$

Since the determinant is non-zero (-96), the vectors are linearly independent.

Conclusion: Do V1, V2, and V3 Span $\mathbb{R}^3$?

Because the determinant of the matrix formed by vectors V1, V2, and V3 is non-zero, these vectors are linearly independent. In linear algebra, three linearly independent vectors in $\mathbb{R}^3$ form a basis for the entire space, meaning they span $\mathbb{R}^3$.

Therefore, V1, V2, and V3 do indeed span $\mathbb{R}^3$.

Additional Insights and Implications

Implication of Linear Independence

- Since the vectors are linearly independent, any vector in $\mathbb{R}^3$ can be expressed uniquely as a linear combination of V1, V2, and V3.
- This makes them a basis for $\mathbb{R}^3$, which is fundamental in many applications such as coordinate transformations, vector space decompositions, and more.

Expressing an Arbitrary Vector as a Linear Combination

Suppose we want to express an arbitrary vector (x, y, z) in $\mathbb{R}^3$ as:

$$
(x, y, z) = c_1 V_1 + c_2 V_2 + c_3 V_3
$$

This leads to the system:

$$
\begin{cases}
0 c_1 + 0 c_2 + 6 c_3 = x \\
0 c_1 + 4 c_2 + 5 c_3 = y \\
4 c_1 + 12 c_2 + 8 c_3 = z
\end{cases}
$$

which can be solved for \(c_1, c_2, c_3 \) for any given \((x, y, z) \).

Summary

- The vectors V1 = (0, 0, 4), V2 = (0, 4, 12), and V3 = (6, 5, 8) form a set of three vectors in $\mathbb{R}^3$.
- Calculating the determinant of the matrix of these vectors yields a non-zero value (-96), indicating the vectors are linearly independent.
- Because they are linearly independent and there are three vectors in $\mathbb{R}^3$, they form a basis for the space.
- Consequently, these vectors span $\mathbb{R}^3$.

In conclusion, V1, V2, and V3 do span the entire three-dimensional space.

Additional Resources for Further Study

- Linear Algebra Textbooks: To deepen your understanding of span, basis, and linear independence.
- Online Calculators: For computing determinants, solving systems of equations, and analyzing vector spaces.
- Linear Algebra Courses: Many free courses are available on platforms like Khan Academy, Coursera, and edX.

By mastering these concepts, you can analyze any set of vectors to determine their span, independence, and role within vector spaces, which is essential in advanced mathematics, physics, computer science, and engineering.

Frequently Asked Questions

Do the vectors V1= (0, 0, 4), V2= (0, 4, 12), and V3= (6, 5, 8) span R^3?

To determine if they span R^3, we need to check if they are linearly independent. Since there are three vectors in 3D space, if they are linearly independent, they span R^3. Calculating the determinant of the matrix formed by these vectors will help; if the determinant is non-zero, they span R^3. In this case, the vectors are linearly dependent because V3 can be expressed as a linear combination of V1 and V2.

How can we verify if the vectors V1, V2, and V3 span 3-dimensional space?

We verify whether the vectors are linearly independent by checking if the determinant of the matrix formed by them is non-zero. If the determinant is zero, they do not span R^3; if it's non-zero, they do.

Are the vectors V1= (0, 0, 4), V2= (0, 4, 12), V3= (6, 5, 8) linearly independent?

No, these vectors are linearly dependent because V3 can be written as a linear combination of V1 and V2, indicating they do not form a basis for R^3.

What is the reason that V1, V2, and V3 do not span R^3?

Because the vectors are linearly dependent, meaning one can be expressed as a combination of the others, they do not span the entire 3D space.

Can V3 be written as a linear combination of V1 and V2?

Yes, V3 can be expressed as a linear combination of V1 and V2, which confirms their linear dependence and that they do not span R^3.

How do we determine if a set of vectors spans 3D space?

By checking if the vectors are linearly independent; this can be done by computing the determinant of the matrix formed by the vectors. A non-zero determinant indicates they span R^3.

What steps should be taken to check if V1, V2, and V3 span 3D space?

Arrange the vectors as columns in a matrix, compute its determinant, and see if it is non-zero. If non-zero, they span R^3; if zero, they do not.

What is the significance of the determinant being zero for these vectors?

A zero determinant indicates the vectors are linearly dependent, meaning they do not form a basis for R^3 and thus do not span the entire space.

Based on the vectors provided, what is the conclusion about their spanning capability?

Since V3 can be expressed as a linear combination of V1 and V2, the vectors are linearly dependent and do not span R^3.

Additional Resources

Does V1, V2, V3 Span $\mathbb{R}^3$? An In-Depth Investigation

In the realm of linear algebra, understanding the span of a set of vectors is fundamental to grasping the structure of vector spaces. When given a collection of vectors, a natural question arises: do these vectors span the entire space? This inquiry becomes especially pertinent when considering three vectors in $\mathbb{R}^3$, where the potential for spanning the entire space hinges on their linear independence and the rank of the matrix they form.

This article embarks on a comprehensive analysis of whether the vectors $V_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$, and $V_3 = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$ span $\mathbb{R}^3$. We will explore the underpinning theoretical concepts, perform systematic calculations, and interpret the results to answer this pivotal question.

Understanding the Concept of Span and Its Significance

Before diving into calculations, it's essential to clarify what "span" means in the context of vector spaces.

Definition of Span

The span of a set of vectors $\{V_1, V_2, V_3\}$ in $\mathbb{R}^3$ is the collection of all possible linear combinations of these vectors:

$$
\text{Span}\{V_1, V_2, V_3\} = \left\{ \alpha V_1 + \beta V_2 + \gamma V_3 \mid \alpha, \beta, \gamma \in \mathbb{R} \right\}
$$

If the span equals $\mathbb{R}^3$, then the vectors are said to span the space, meaning any vector in $\mathbb{R}^3$ can be expressed as a linear combination of V_1, V_2, V_3.

Why Is Determining the Span Important?

- Basis Formation: If the vectors span $\mathbb{R}^3$ and are linearly independent, they form a basis for the space.

- Dimensionality & Independence: The span reveals whether vectors are independent or redundant.
- Applications: From solving systems of equations to understanding subspaces, the concept underpins many areas in mathematics and engineering.

Step-by-Step Analysis of the Given Vectors

Given vectors:

$$
V_{1} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad V_{2} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}, \quad V_{3} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}
$$

Our goal is to determine whether these vectors span $\mathbb{R}^3$. The most straightforward method involves examining their linear independence, typically through constructing a matrix with these vectors as columns and analyzing its rank.

Constructing the Matrix and Computing Its Rank

Forming the Matrix

Arrange the vectors as columns in a matrix A:

$$
A = \begin{bmatrix}
0 & 0 & 6 \\
0 & 4 & 5 \\
4 & 12 & 8
\end{bmatrix}
$$

The rank of A indicates the dimension of the span:

- Rank 3: vectors span $\mathbb{R}^3$.

- Rank less than 3: vectors do not span \(\mathbb{R}^3\).

Calculating the Rank via Row Reduction

Perform Gaussian elimination to find the rank.

Step 1: Write the augmented matrix:

\[
\begin{bmatrix}
0 & 0 & 6 \\
0 & 4 & 5 \\
4 & 12 & 8
\end{bmatrix}
\]

Step 2: Swap rows to bring a non-zero pivot to the top if necessary.

- The first row's first element is zero; swap Row 1 and Row 3:

\[
\begin{bmatrix}
4 & 12 & 8 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{bmatrix}
\]

Step 3: Use Row 1 to eliminate entries below:

- The first pivot is 4 in position (1,1).

- Rows 2 and 3 have zeros in the first column; no elimination needed there.

Step 4: Normalize Row 1:

\[
R_1 \leftarrow \frac{1}{4} R_1 \Rightarrow \begin{bmatrix}
1 & 3 & 2 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{bmatrix}

\]

Step 5: Use Row 1 to eliminate entries in other rows (here, no effect as rows 2 and 3 are zeros in first column).

Step 6: Normalize Row 2:

\[
R_2 \leftarrow \frac{1}{4} R_2 \Rightarrow \begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & \frac{5}{4} \\
0 & 0 & 6
\end{bmatrix}
\]

Step 7: Normalize Row 3:

\[
R_3 \leftarrow \frac{1}{6} R_3 \Rightarrow \begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & \frac{5}{4} \\
0 & 0 & 1
\end{bmatrix}
\]

Now, the matrix is in row echelon form.

Analyzing the Results of Row Reduction

The row echelon form reveals:

- Three non-zero rows.
- Therefore, the rank of matrix \(A\) is 3.

Since the rank equals the number of vectors (3), and the vectors are in \(\mathbb{R}^3\), this indicates that:

The vectors \(V_1, V_2, V_3\) are linearly independent and span \(\mathbb{R}^3\).

Verifying Linear Independence Directly

While the row reduction suggests independence, it's instructive to verify explicitly whether the vectors satisfy any linear dependence relation.

Suppose:

$$
\alpha V_1 + \beta V_2 + \gamma V_3 = 0
$$

This translates to the system:

$$
\alpha \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix} + \gamma \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Leading to the equations:

1. $0\alpha + 0\beta + 6\gamma = 0 \Rightarrow 6\gamma = 0 \Rightarrow \gamma=0$

2. $0\alpha + 4\beta + 5\gamma=0 \Rightarrow 4\beta=0 \Rightarrow \beta=0$

3. $4\alpha + 12\beta + 8\gamma=0 \Rightarrow 4\alpha=0 \Rightarrow \alpha=0$

All coefficients are zero, confirming that the vectors are linearly independent.

Implications of the Analysis

Given the complete analysis above, the key conclusion is:

- The vectors V_1, V_2, V_3 form a set of linearly independent vectors in $\mathbb{R}^3$.
- The rank of the matrix formed by these vectors is 3.
- As a result, the vectors span the entire $\mathbb{R}^3$.

Therefore, V_1, V_2, V_3 do span $\mathbb{R}^3$.

Additional Observations and Insights

While the calculation confirms the spanning property, it's worth noting some subtleties and potential misconceptions:

- Non-zero vectors are not always sufficient: The vectors must also be linearly independent.
- Dependence on the matrix structure: The specific arrangement and entries matter; for example, if one vector is a linear combination of the others, they won't span $\mathbb{R}^3$.

Interestingly, although V_1 and V_2 are related (since $V_2= \frac{1}{3} V_{1} \times 3$) is not true, but observing their entries shows that V_2 is not a scalar

[Let V1 0 0 4 V2 0 4 12 And V3 6 5 8 Does V1 V2 V3 Span 3 Why Or Why Not]

Let V1 0 0 4 V2 0 4 12 And V3 6 5 8 Does V1 V2 V3 Span 3 Why Or Why Not

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