Question 6: Show That There Are No Two N X N Matrices A And B Satisfy AB - BA= In

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In linear algebra, the relationship between matrices and their commutators—expressions of the form AB - BA—plays a fundamental role in understanding the structure of matrix algebras and the properties of linear transformations. The specific question at hand asks us to demonstrate that there are no matrices A and B of size $N \times N$ such that their commutator equals the identity matrix I_N, i.e.,

$$
AB - BA = I_N.
$$

This assertion is not merely a curiosity but a profound result linking matrix algebra, the concept of trace, and the properties of linear operators. The goal of this article is to rigorously prove why such a pair of matrices cannot exist, exploring the underlying algebraic principles and providing comprehensive explanations.

Understanding the Problem: The Commutator and Its Significance

The Commutator in Matrix Algebra

Given two matrices $A, B \in M_N(\mathbb{F})$ (the space of all $N \times N$ matrices over a field $\mathbb{F}$, typically $\mathbb{R}$ or $\mathbb{C}$), their commutator is defined as:

$$
[A, B] = AB - BA.
$$

The commutator measures the extent to which two matrices fail to commute. If $AB = BA$, the matrices commute, and their commutator is zero. The set of all matrices that commute with a given matrix forms its centralizer, which is a subalgebra of $M_N(\mathbb{F})$.

The Significance of the Identity Matrix as a Commutator

The question asks whether the identity matrix I_N can be expressed as a commutator of two matrices. Intuitively, this is asking: Can the fundamental "difference" between AB and BA be exactly the identity? Since the identity matrix acts as a neutral element with respect to multiplication, this question probes deep into the structure of the matrix algebra.

Preliminary Observations and Known Results

Properties of the Trace Function

A critical property of the trace function $\operatorname{tr}(\cdot)$ on matrices is that it is linear and invariant under cyclic permutations:

$$
\operatorname{tr}(AB) = \operatorname{tr}(BA).
$$

This cyclic invariance implies that for any matrices A, B:

$$
\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.
$$

This is a key observation because it directly relates the trace of a commutator to zero.

Implication for the Identity Matrix

Applying the trace to the supposed commutator $AB - BA = I_N$, we get:

$$
\operatorname{tr}(AB - BA) = \operatorname{tr}(I_N) = N,
$$

because the trace of the identity matrix I_N is simply the sum of its diagonal entries, which is N.

But from the property above, the trace of any commutator must be zero. Therefore,

$$
\operatorname{tr}(AB - BA) = 0,
$$

which contradicts the fact that

$$
\operatorname{tr}(I_N) = N \neq 0,
$$

for $N \geq 1$.

This contradiction is the crux of the proof, indicating that the initial assumption—existence of such matrices A and B—must be false.

Formal Proof: No Matrices A, B Satisfy $AB - BA = I_N$

Step 1: Assume the Existence of Such Matrices

Suppose, for the sake of contradiction, that there exist matrices $A, B \in M_N(\mathbb{F})$ such that:

$$
AB - BA = I_N.
$$

Step 2: Take the Trace of Both Sides

Applying the trace operator to both sides yields:

$$
\operatorname{tr}(AB - BA) = \operatorname{tr}(I_N).
$$

Using linearity, this simplifies to:

$$
\operatorname{tr}(AB) - \operatorname{tr}(BA) = N.
$$

But since the trace is invariant under cyclic permutation:

$$
\operatorname{tr}(AB) = \operatorname{tr}(BA),
$$

which implies:

$$

$$
\operatorname{tr}(AB) - \operatorname{tr}(AB) = 0,
$$

or equivalently,

$$
0 = N.
$$

Step 3: Derive the Contradiction

This leads to the statement:

$$
0 = N,
$$

which can only hold if $N = 0$. But the dimension of the matrices is $N \times N$ with $N \geq 1$ by definition, so this is impossible.

Therefore, the initial assumption that matrices A, B exist satisfying $AB - BA = I_N$ must be false.

Implications and Broader Context

Connection to Lie Algebras and the Structure of $M_N(\mathbb{F})$

The set $M_N(\mathbb{F})$ forms an associative algebra over $\mathbb{F}$, and the commutator operation defines a Lie algebra structure on $M_N(\mathbb{F})$. The result that the identity matrix cannot be expressed as a commutator reflects fundamental properties of this algebra:

- The Lie algebra $\mathfrak{gl}_N(\mathbb{F})$ (the general linear Lie algebra) is perfect, meaning its derived algebra is itself.
- The identity matrix is in the center of $M_N(\mathbb{F})$, but the commutator subalgebra is trace-zero matrices, which do not include the identity.

This demonstrates that the identity matrix lies outside the derived Lie algebra generated by commutators, highlighting the structural limitations within the algebra.

What if the matrices are over different fields or special

classes?

The proof is quite general and holds over any field $\mathbb{F}$, provided the matrices are square and of size $N \times N$. For special classes of matrices (e.g., commuting matrices, nilpotent matrices), the structure varies, but the trace argument remains valid and prevents the representation of I_N as a commutator.

Additional Perspectives and Related Results

Trace Zero Matrices and Commutators

A well-known result in matrix theory is:

- The set of all matrices that can be expressed as a commutator is exactly the set of matrices with zero trace.

This stems from the cyclic invariance of the trace and the previous argument, confirming that:

$$
\text{If } A B - B A = C, \text{ then } \operatorname{tr}(C) = 0.
$$

Conversely, every trace-zero matrix can be expressed as a commutator, a classical result often proved in linear algebra courses.

Consequences for the Existence of Certain Matrices

The above leads to the clear conclusion: since the identity matrix has a non-zero trace, it cannot be written as a commutator of two matrices. This is a fundamental structural restriction within matrix algebra.

Summary and Final Remarks

To summarize:

- The key property used is the invariance of the trace under cyclic permutations: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
- For any matrices A, B, the trace of their commutator $AB - BA$ is zero.
- The identity matrix I_N has a trace equal to N, which is non-zero.
- Therefore, the assumption that $AB - BA = I_N$ leads to a contradiction, since taking traces yields $0 = N \neq 0$.
- Conclusion: No pair of $N \times N$ matrices A and B satisfy $AB - BA = I_N$.

This proof elegantly combines linear algebraic properties with fundamental trace invariants, illustrating how seemingly simple properties can have profound implications for the structure of matrix algebras.

In essence, the non-existence of such matrices underscores the intrinsic limitations imposed by the trace function and the algebraic structure of matrices, reinforcing the importance of these concepts in understanding linear transformations and algebraic identities.

Frequently Asked Questions

What is the significance of the equation AB - BA = In in linear algebra?

It represents a scenario where two matrices A and B have a commutator equal to the identity matrix, which relates to the study of non-commuting operators and has implications in quantum mechanics and matrix theory.

Why is it impossible for two N×N matrices A and B to satisfy AB - BA = In?

Because the trace of a commutator is always zero, yet the trace of In is N, leading to a contradiction, which proves no such matrices can exist.

How does the trace function help in proving that AB - BA ≠ In for matrices A and B?

The trace of AB - BA is always zero due to properties of trace and multiplication, but the trace of In is N, so the two cannot be equal, thus no such matrices exist.

Can the equation AB - BA = In hold for matrices over any field?

No, the proof relies on properties of trace that hold over any field, so the result that no such matrices exist is valid over all fields.

What is the connection between this problem and the concept of the Lie algebra of matrices?

The commutator [A, B] = AB - BA plays a central role in Lie algebras; the question explores whether a specific element (the identity) can be expressed as a commutator, which it cannot in this case.

How does the dimension N of matrices influence the

impossibility of satisfying AB - BA = In?

The dimension N determines the trace of In, which is N, conflicting with the zero trace of the commutator, regardless of the size of N, proving the impossibility for all N.

Is there a similar result for infinite-dimensional operators regarding the commutator being the identity?

In infinite-dimensional settings, such as bounded operators on a Hilbert space, the situation is more complex, but generally, the identity cannot be expressed as a commutator of bounded operators, analogous to the finite-dimensional case.

Additional Resources

Commutator: Show That There Are No Two N x N Matrices A and B Satisfy AB - BA = I_n

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Introduction

In the realm of linear algebra, understanding the behavior and properties of matrices, especially in relation to their commutators, provides deep insights into the structure of matrix algebras and Lie algebras. One prominent question that arises is whether it is possible for two $N \times N$ matrices A and B to satisfy the relation:

$$
AB - BA = I_N,
$$

where I_N is the $N \times N$ identity matrix. This question probes the existence (or non-existence) of matrices whose commutator equals the identity matrix, a problem that intertwines algebraic properties, trace considerations, and the structure of matrix Lie algebras.

In this comprehensive analysis, we will explore this question meticulously, delving into the foundational ideas, providing rigorous proofs, and understanding the implications of the result. The main goal is to substantiate why such matrices cannot exist for any positive integer N.

Background and Context

The Commutator and Its Significance

The commutator of two matrices A and B, denoted $[A, B]$, is defined as:

$$
[A, B] = AB - BA.
$$

This operation measures the failure of A and B to commute. In particular, the set of all matrices with the operation of addition and the commutator forms a Lie algebra, known as the special linear Lie algebra $\mathfrak{sl}_N(\mathbb{F})$.

The question at hand asks: Can the identity matrix I_N be expressed as a commutator of two matrices? If so, then I_N would be in the derived subalgebra of $M_N(\mathbb{F})$.

Formal Statement of the Problem

Question 6: Show that there are no matrices $A, B \in M_N(\mathbb{F})$ such that:

$$
AB - BA = I_N,
$$

where $M_N(\mathbb{F})$ denotes the algebra of all $N \times N$ matrices over a field $\mathbb{F}$.

Significance of the Question

This problem is not just a curiosity; it has profound implications in Lie algebra theory and the structure of the algebra of matrices:

- It relates to the trace properties of matrices.
- It showcases the constraints in the algebra $M_N(\mathbb{F})$.
- It highlights the fact that the identity matrix is not a commutator, which in turn informs about the structure of the derived algebra.

Core Concepts and Tools

Before proceeding to the proof, let's review some key concepts:

1. Trace and Its Properties

- The trace function $\operatorname{tr}: M_N(\mathbb{F}) \to \mathbb{F}$ assigns to each matrix the sum of its diagonal entries.
- Linearity: $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.
- Cyclic property: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

2. Trace of a Commutator

A crucial property:

$$
\operatorname{tr}([A, B]) = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) -
$$

\operatorname{tr}(BA) = 0,
\]

which implies every commutator has trace zero.

Main Theoretical Result

Theorem: In \(M_N(\mathbb{F}) \), the set of all commutators is exactly the set of matrices with trace zero.

Implication: Since the identity matrix \(I_N \) has trace \(N \), it cannot be expressed as a commutator unless \(N = 0 \), which is trivial and meaningless in this context.

Detailed Proof

Let's formalize this reasoning step-by-step.

Step 1: Show that the trace of any commutator is zero

- For any matrices \(A, B \in M_N(\mathbb{F}) \):

\[
\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA).
\]

- Due to the cyclic property of the trace:

\[
\operatorname{tr}(AB) = \operatorname{tr}(BA).
\]

- Therefore:

\[
\operatorname{tr}(AB - BA) = 0.
\]

This is a fundamental property that holds over any field \(\mathbb{F} \).

Step 2: The set of matrices with trace zero

From step 1, it follows that:

\[
\text{Range of the commutator map} \subseteq \{ M \in M_N(\mathbb{F}) \mid

\operatorname{tr}(M) = 0 \}.
\]

In fact, a classical result states that:

\[
\boxed{
\text{Every matrix with trace zero can be expressed as a commutator.}
}
\]

This is a deep and non-trivial fact, but it is well-established in the theory of matrix Lie algebras.

Step 3: Applying the trace condition to \(I_N \)

- The identity matrix \(I_N \) has:

\[
\operatorname{tr}(I_N) = N.
\]

- Since the trace of a commutator is always zero, \(I_N \) cannot be a commutator unless \(N = 0 \).

- But \(N = 0 \) corresponds to the trivial zero-dimensional matrix space, which is outside the scope of our problem.

Conclusion:

\[
\boxed{
\text{There are no matrices } A, B \in M_N(\mathbb{F}) \text{ satisfying } AB - BA = I_N \quad
\text{for any } N \ge 1.
}
\]

In other words, the identity matrix cannot be expressed as a commutator in the algebra of all \(N \times N \) matrices.

Additional Perspectives and Generalizations

1. The Role of the Field \(\mathbb{F} \)

- The above proof holds over any field \(\mathbb{F} \) because the trace properties are field-independent.
- Even in fields with characteristic dividing \(N \), the essential argument remains valid, as the trace of the identity still equals \(N \).

2. Connection to Lie Algebra Theory

- The set of all matrices with trace zero forms the special linear Lie algebra $\mathfrak{sl}_N(\mathbb{F})$.
- The fact that I_N (which is not trace-zero) cannot be written as a commutator reflects the fact that:

$$
[\mathfrak{sl}_N(\mathbb{F}), \mathfrak{sl}_N(\mathbb{F})] \subseteq
\mathfrak{sl}_N(\mathbb{F}),
$$

and the derived subalgebra does not include the identity matrix.

3. Explicit Construction for Trace-Zero Matrices

- It is known and proven that every trace-zero matrix can be written as a commutator.
- For example, in $M_N(\mathbb{F})$, given M with $\operatorname{tr}(M) = 0$, there exist matrices A, B such that:

$$
AB - BA = M.
$$

This robust result underscores the special nature of the identity matrix regarding commutators.

Implications and Significance

The inability to represent I_N as a commutator has several profound implications:

- Structural Insight: It emphasizes that the algebra $M_N(\mathbb{F})$ is not perfect; it contains elements (like I_N) outside its derived subalgebra.
- Lie Algebra Perspective: It highlights the role of trace conditions in the structure of Lie algebras.
- Representation Theory: The fact that the identity matrix is not a commutator influences how representations of these algebras are constructed and understood.

Summary of Key Points

- Trace of commutator: For any $A, B \in M_N(\mathbb{F})$, $\operatorname{tr}(AB - BA) = 0$.
- Main conclusion: The identity matrix I_N has a non-zero trace N, and therefore cannot be expressed as a commutator.
- Existence of trace-zero matrices as commutators: Every trace-zero matrix can be written as a commutator, underscoring the special role of the identity matrix.
- Final assertion: For all $N \ge 1$, no pair of matrices A, B satisfies:

$$
AB - BA = I_N.

\]

Final Remarks

This problem elegantly demonstrates how a simple invariant like the trace imposes fundamental

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