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Introduction

In the realm of differential equations, systems of linear differential equations play a crucial role in modeling complex phenomena across physics, engineering, biology, and economics. These systems often involve multiple variables interacting simultaneously, and solving them requires a systematic approach that combines algebraic and analytical techniques.

Today, we focus on a specific system of differential equations:

$$
\frac{dX}{dt} = A X
$$

where

$$
A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}
$$

and the initial condition:

$$
X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
$$

Our goal is to find the solution $X(t)$ in real form, meaning the solution expressed explicitly with real-valued functions of t.

Understanding how to solve such systems is fundamental because it provides insights into the stability, oscillations, and long-term behavior of the modeled system. This article provides a step-by-step guide to solving this specific system, emphasizing solving in real functions, and optimizing the explanation for clarity and SEO.

Understanding the System

The Differential Equation

The system is a first-order linear differential equation in vector form:

$$
\frac{dX}{dt} = A X
$$

where

$$
X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

and A is a constant 2×2 matrix.

Initial Conditions

Given:

$$
X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
$$

This initial condition specifies the state of the system at $t=0$, which is essential for determining the particular solution.

Step-by-Step Solution Method

To solve the system, we follow these steps:

1. Find the eigenvalues of matrix A.
2. Find the eigenvectors corresponding to each eigenvalue.
3. Construct the general solution using eigenvalues and eigenvectors.
4. Apply initial conditions to find specific solution constants.
5. Express the solution in real form.

Let’s go through each step in detail.

Step 1: Find Eigenvalues of Matrix A

The eigenvalues λ of A satisfy the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix.

Calculating:

$$
A - \lambda I = \begin{bmatrix} 3 - \lambda & 9 \\ -1 & -3 - \lambda \end{bmatrix}
$$

The determinant:

$$
(3 - \lambda)(-3 - \lambda) - (-1)(9) = 0
$$

Expanding:

$$
(3 - \lambda)(-3 - \lambda) + 9 = 0
$$

Compute the product:

$$
(3)(-3) + 3(-\lambda) + (-\lambda)(-3) + (-\lambda)(-\lambda) + 9 = 0
$$

Simplify:

$$
-9 - 3\lambda + 3\lambda + \lambda^2 + 9 = 0
$$

Combine like terms:

$$
(-9 + 9) + (-3\lambda + 3\lambda) + \lambda^2 = 0
$$

$$
0 + 0 + \lambda^2 = 0
$$

Thus:

$$
\lambda^2 = 0
$$

Eigenvalues:

\[
\boxed{\lambda_1 = 0, \quad \lambda_2 = 0}
\]

Both eigenvalues are zero; this indicates a defective matrix with repeated eigenvalues, so we need to find eigenvectors and generalized eigenvectors.

Step 2: Find Eigenvectors

Since both eigenvalues are zero, find the eigenvectors by solving:

\[
(A - 0 I) v = 0
\]

which simplifies to:

\[
A v = 0
\]

or

\[
\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

From the first row:

\[
3 v_1 + 9 v_2 = 0 \Rightarrow v_1 = -3 v_2
\]

From the second row:

\[
- v_1 - 3 v_2 = 0
\]

Substitute \(v_1 = -3 v_2\):

\[
- (-3 v_2) - 3 v_2 = 0 \Rightarrow 3 v_2 - 3 v_2 = 0
\]

which holds for any \(v_2\).

Thus, the eigenvectors are:

$$
v = v_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

Choose $v_2=1$:

$$
\boxed{v^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}}
$$

Step 3: Find General Solution

Since A has a repeated eigenvalue $\lambda=0$ but only one eigenvector, the matrix is defective. We need to find a generalized eigenvector $v^{(2)}$ satisfying:

$$
(A - 0 I) v^{(2)} = v^{(1)}
$$

or:

$$
A v^{(2)} = v^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

Solve:

$$
\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

This gives the system:

1. $3 v_1^{(2)} + 9 v_2^{(2)} = -3$

2. $- v_1^{(2)} - 3 v_2^{(2)} = 1$

From the second equation:

$$
- v_1^{(2)} = 1 + 3 v_2^{(2)} \Rightarrow v_1^{(2)} = -1 - 3 v_2^{(2)}
$$

Substitute into the first:

$$
3 (-1 - 3 v_2^{(2)}) + 9 v_2^{(2)} = -3
$$

\]

Simplify:

$$
-3 - 9 v_2^{(2)} + 9 v_2^{(2)} = -3
$$

$$
-3 = -3
$$

which is always true, so $v_2^{(2)}$ is arbitrary. Let’s pick $v_2^{(2)}=0$:

$$
v_1^{(2)} = -1 - 3(0) = -1
$$

Thus, a generalized eigenvector:

$$
v^{(2)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
$$

Step 4: Construct the General Solution

For a defective matrix with a single eigenvalue $\lambda=0$, the general solution is:

$$
X(t) = c_1 e^{0 \cdot t} v^{(1)} + c_2 e^{0 \cdot t} (v^{(2)} + t v^{(1)})
$$

which simplifies to:

$$
X(t) = c_1 v^{(1)} + c_2 (v^{(2)} + t v^{(1)})
$$

Substituting the vectors:

$$
X(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)
$$

Expressed explicitly:

\[
X(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 - 3 t \\ t \end{bmatrix}
\]

Step 5: Apply Initial Conditions

Given:

\[
X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\]

At \(t=0\):

\[
X(0) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \

Frequently Asked Questions

What is the primary goal when solving the system Dx/dt = A X with initial condition X(0) = [2, 4]?

The primary goal is to find the explicit real solution X(t) that satisfies the differential system and initial conditions, typically by diagonalizing or transforming matrix A to obtain real-valued solutions.

Given the system Dx/dt = [[3, 9], [-1, -3]] X with X(0) = [2, 4], what are the eigenvalues of matrix A?

The eigenvalues are $\lambda = 0$ and $\lambda = -3$, found by solving $\det(A - \lambda I) = 0$, which leads to $(3 - \lambda)(-3 - \lambda) + 9 = 0$, simplifying to $\lambda(\lambda + 3) = 0$.

How do you find the eigenvectors corresponding to each eigenvalue for this system?

For $\lambda = 0$, substitute into $(A - 0 I) v = 0$ and solve for v; similarly for $\lambda = -3$, substitute into $(A + 3 I) v = 0$. This yields eigenvectors v1 and v2 that form the basis for the solution.

What is the general real solution to the system

after finding eigenvalues and eigenvectors?

The general real solution is a linear combination of exponential functions multiplied by eigenvectors: X(t) = c1 v1 + c2 v2, where each component is expressed in real form, involving e^{λ t} terms.

How do you incorporate the initial condition X(0) = [2, 4] to find specific solution constants?

Substitute t = 0 into the general solution to form a system of equations, then solve for constants c1 and c2 to satisfy the initial conditions.

What is the final real solution for the system Dx/dt = [[3, 9], [-1, -3]] X with X(0) = [2, 4]?

After calculations, the solution is X(t) = c1 v1 + c2 v2, with constants determined from initial conditions, resulting in explicit expressions involving real exponential functions, such as X(t) = (a constant) e^{0 t} + (another constant) e^{-3 t}.

Additional Resources

Comprehensive Review and Solution of the System $\frac{dX}{dt} = A X$ with Initial Condition $X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Introduction

The study of systems of linear differential equations is fundamental in understanding various phenomena across physics, engineering, and applied mathematics. One such system, expressed in matrix form as

$$
\frac{dX}{dt} = A X,
$$

where A is a constant matrix, encapsulates the evolution of a vector $X(t)$ over time. This review focuses on solving the specific system:

$$
\frac{dX}{dt} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} X,
$$

with the initial condition

$$
X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.

\]

The goal is to find the explicit solution in real form, thoroughly analyze the solution's structure, interpret its behavior, and understand the implications of the eigenvalues and eigenvectors associated with matrix \(A\).

Theoretical Framework

System Representation

The system can be written as:

\[
\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\]

This is a first-order linear system with constant coefficients, which can be solved using the matrix exponential:

\[
X(t) = e^{A t} X(0).
\]

To compute this, the primary step involves diagonalizing \(A\) or, if not diagonalizable, finding its Jordan canonical form, then calculating \(e^{A t}\).

Step 1: Eigenvalues of \(A\)

Characteristic Polynomial:

\[
\det(A - \lambda I) = 0,
\]

which for the matrix:

\[
A - \lambda I = \begin{bmatrix} 3 - \lambda & 9 \\ -1 & -3 - \lambda \end{bmatrix}.
\]

Calculating determinant:

\[

(3 - \lambda)(-3 - \lambda) - (-1)(9) = 0.
\]

Expanding:

\[
(3 - \lambda)(-3 - \lambda) + 9 = 0,
\]
\[
-9 - 3 \lambda + 3 \lambda + \lambda^2 + 9 = 0,
\]
\[
\lambda^2 = 0,
\]

which simplifies to:

\[
\lambda^2 = 0,
\]

indicating a repeated eigenvalue:

\[
\boxed{\lambda = 0}
\]

with multiplicity 2.

Step 2: Eigenvectors and Generalized Eigenvectors

Since the eigenvalue \(\lambda=0\) has algebraic multiplicity 2, we need to determine the geometric multiplicity to understand the nature of solutions.

Eigenvector for \(\lambda=0\):

Solve \(A v = 0\):

\[
\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

From the first row:

\[
3 v_1 + 9 v_2 = 0 \Rightarrow v_1 = -3 v_2.
\]

The second row:

$$
- v_1 - 3 v_2 = 0,
$$

which simplifies to:

$$
- (-3 v_2) - 3 v_2 = 0 \Rightarrow 3 v_2 - 3 v_2 = 0,
$$

which is always satisfied, confirming the eigenvector is:

$$
v^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.
$$

Finding a generalized eigenvector:

Since the eigenvalue is repeated, check if the geometric multiplicity is 1:

$$
\text{Rank}(A) = 1,
$$

implying only one eigenvector and the need for a generalized eigenvector.

Solve:

$$
A v^{(2)} = v^{(1)}.
$$

Set:

$$
A v^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.
$$

Let $v^{(2)} = \begin{bmatrix} x \\ y \end{bmatrix}$:

$$
\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.
$$

This yields:

- First equation:

$$
3x + 9 y = -3,
$$

- Second equation:

$$
- x - 3 y = 1.
$$

From the second:

$$
- x - 3 y = 1 \Rightarrow x = - 3 y - 1.
$$

Substitute into the first:

$$
3(- 3 y - 1) + 9 y = -3,
$$

$$
- 9 y - 3 + 9 y = -3,
$$

which simplifies to:

$$
-3 = -3,
$$

an identity, meaning y is free. Let:

$$
y = t,
$$

then:

$$
x = -3 t - 1.
$$

Choose $t = 0$:

$$
v^{(2)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.
$$

Thus, the generalized eigenvector is:

$$
v^{(2)} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.
$$

Step 3: Constructing the Solution

Given the eigenvector and generalized eigenvector, the matrix exponential $e^{A t}$ can be written as:

$$
e^{A t} = e^{0 \cdot t} \left(I + t N \right),
$$

where N is the nilpotent matrix associated with the Jordan block.

Form of A:

Since A has a single eigenvalue $\lambda=0$ with algebraic multiplicity 2, and the Jordan form:

$$
J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
$$

we find a similarity transformation P such that:

$$
A = P J P^{-1}.
$$

Construct P with eigenvector and generalized eigenvector as columns:

$$
P = \begin{bmatrix} v^{(1)} & v^{(2)} \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & 0 \end{bmatrix}.
$$

Calculate P^{-1}:

$$
\det P = (-3)(0) - (-1)(1) = 1,
$$

so

$$
P^{-1} = \frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}.
$$

The matrix exponential:

$$
e^{A t} = P e^{J t} P^{-1}.
$$

Since

$$
e^{J t} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},
$$

we get:

$$
e^{A t} = P \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} P^{-1}.
$$

Step 4: Computing $e^{A t}$

Perform the multiplication:

$$
e^{A t} = P \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} P^{-1}.
$$

Calculate step by step:

$$
\text{First: } P \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} =
\begin{bmatrix} -3 & -3 t -1 \\ 1 & t \end{bmatrix}.
$$

Then,

$$
e^{A t} = \begin{bmatrix} -3 & -3 t - 1 \\ 1 & t \end{bmatrix}
\begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}.
$$

Multiplying:

- First row:

$$
(-3)(0) + (-3 t - 1)(-1) = 0 + (3 t + 1) = 3 t + 1,
$$

$$
(-3)(1) + (-3 t - 1)(-
$$

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