Suppose That We Roll A Fair Die And Let X Be The Resulting Number Find E(x),var(x)

Suppose That We Roll A Fair Die And Let X Be The Resulting Number Find E(x),var(x). This scenario is a classic example in probability theory and statistics, illustrating fundamental concepts such as expected value and variance. Understanding these concepts is essential for analyzing random experiments, assessing uncertainty, and making informed decisions based on probabilistic models. In this article, we will explore the mathematical foundations behind rolling a fair six-sided die, how to compute the expected value (E(x)), the variance (var(x)), and interpret their significance within the context of probability distributions.

Understanding the Basics of a Fair Die

What Is a Fair Die?

A fair die is a six-sided cube with faces numbered from 1 to 6, where each face has an equal probability of landing face-up when rolled. The fairness of the die ensures that no outcome is favored over another, making the probability of each result exactly 1/6. This symmetry makes the die an ideal model for studying discrete uniform probability distributions.

Sample Space and Outcomes

The sample space (denoted as S) for a single roll of a fair die is:

- S = {1, 2, 3, 4, 5, 6}

Each outcome in this space has an equal probability:

- P(X = x) = 1/6 for x = 1, 2, 3, 4, 5, 6

Defining the Random Variable X

What Is a Random Variable?

A random variable (denoted as X) is a function that assigns a real number to each outcome in the sample space. In this case, X maps each die roll to its corresponding number:

- X(roll) = the number showing on the top face after a die is rolled

Probability Distribution of X

Since the die is fair, the probability distribution of X is uniform across its possible values:

- $P(X = x) = 1/6$ for $x = 1, 2, 3, 4, 5, 6$

This distribution is called a discrete uniform distribution because each outcome has the same probability.

Calculating the Expected Value E(x)

Definition of Expected Value

The expected value (or mean) of a discrete random variable is a measure of the central tendency, representing the average outcome if the experiment is repeated many times. It is mathematically expressed as:

- $E(X) = \Sigma [x P(X = x)]$

where the summation runs over all possible values of X.

Calculating E(x) for a Fair Die

Given the probability distribution:

- $P(X = x) = 1/6$ for $x = 1, 2, 3, 4, 5, 6$

we compute:

- $E(X) = (1)(1/6) + (2)(1/6) + (3)(1/6) + (4)(1/6) + (5)(1/6) + (6)(1/6)$

which simplifies to:

- $E(X) = (1 + 2 + 3 + 4 + 5 + 6) / 6$

Calculating the numerator:

- 1 + 2 + 3 + 4 + 5 + 6 = 21

Therefore:

- E(X) = 21 / 6 = 3.5

Interpretation: The expected value of rolling a fair die is 3.5, which means that over a large number of rolls, the average outcome tends toward 3.5, even though it’s impossible to actually roll a 3.5. This value indicates the long-term average.

Calculating the Variance var(x)

What Is Variance?

Variance measures the spread or dispersion of the outcomes around the expected value. It quantifies how much the values of X deviate from E(X). The formula for variance is:

- var(X) = E[(X - E(X))^2] = Σ [(x - E(X))^2 P(X = x)]

Calculating var(x) for a Fair Die

Step 1: Compute the squared deviations:

- (x - E(X))^2 for each x

Using E(X) = 3.5, the squared deviations are:

- (1 - 3.5)^2 = (-2.5)^2 = 6.25
- (2 - 3.5)^2 = (-1.5)^2 = 2.25
- (3 - 3.5)^2 = (-0.5)^2 = 0.25
- (4 - 3.5)^2 = (0.5)^2 = 0.25
- (5 - 3.5)^2 = (1.5)^2 = 2.25
- (6 - 3.5)^2 = (2.5)^2 = 6.25

Step 2: Multiply each squared deviation by its probability (which is 1/6):

- 6.25 1/6 ≈ 1.0417
- 2.25 1/6 ≈ 0.375
- 0.25 1/6 ≈ 0.0417
- 0.25 1/6 ≈ 0.0417
- 2.25 1/6 ≈ 0.375
- 6.25 1/6 ≈ 1.0417

Step 3: Sum these values to find the variance:

- var(X) ≈ 1.0417 + 0.375 + 0.0417 + 0.0417 + 0.375 + 1.0417 ≈ 2.9167

Result: The variance of the die roll is approximately 2.92.

Significance of E(x) and var(x)

Expected Value as a Measure of Center

The expected value provides a benchmark for the average outcome of the die roll over numerous trials. It helps in understanding the long-term behavior of the random process and is fundamental in decision-making processes involving probabilities.

Variance as a Measure of Spread

Variance indicates how much the results fluctuate around the mean. A higher variance signifies more variability, while a lower variance suggests outcomes are more tightly clustered around the expected value.

Applications and Broader Context

Real-World Applications

Understanding the expected value and variance of simple experiments like rolling a die has broad applications:

- Board games and gambling: calculating expected winnings or losses.

- Risk assessment: measuring variability in outcomes.
- Simulation modeling: predicting average results over multiple trials.
- Quality control: assessing process variability.

Extensions to Other Discrete Distributions

The principles used here extend to other discrete probability distributions, such as binomial, geometric, and Poisson distributions. Each has its own formulas for expected value and variance, but the core concepts remain consistent.

Conclusion

Rolling a fair die and analyzing the resulting number through the lens of expected value and variance offers a clear illustration of fundamental probability concepts. The expected value of 3.5 reflects the average outcome over many rolls, while the variance of approximately 2.92 measures the spread of possible results around this average. These calculations are not only theoretical exercises but also practical tools in fields ranging from statistics and economics to engineering and game theory. Mastery of these concepts equips individuals with the ability to model, analyze, and interpret random phenomena effectively.

Summary:
- The expected value of a fair die roll is 3.5.
- The variance of the die roll is approximately 2.92.
- These measures help understand the central tendency and variability of outcomes.
- The principles extend to more complex probabilistic models and real-world applications.

By grasping these fundamental statistical measures, you can better appreciate the behavior of random variables and their significance in various domains of science and everyday life.

Frequently Asked Questions

What is the expected value E(X) when rolling a fair six-sided die?

The expected value E(X) is 3.5, calculated as (1+2+3+4+5+6)/6.

How do you calculate the variance Var(X) for a fair die

roll?

Variance Var(X) is calculated as the average of the squared deviations from the mean: Var(X) = E[X^2] - (E[X])^2.

What is the value of E[X^2] for a fair six-sided die?

E[X^2] = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)/6 = (1+4+9+16+25+36)/6 = 91/6.

Calculate the variance Var(X) for the result of a fair die roll.

Var(X) = E[X^2] - (E[X])^2 = (91/6) - (3.5)^2 = (91/6) - 12.25 ≈ 15.17 - 12.25 = 2.92.

Why is the expected value of a fair die roll 3.5?

Because the average of all possible outcomes (1 through 6) is (1+2+3+4+5+6)/6 = 3.5.

Is the variance of a fair die roll symmetric around the mean?

Yes, since each outcome is equally likely, the variance reflects the spread of outcomes around the mean 3.5.

What assumptions are made in calculating E(X) and Var(X) for a die roll?

The main assumption is that the die is fair, meaning each outcome from 1 to 6 has an equal probability of 1/6.

How would the calculations change if the die were biased?

If the die were biased, you'd need the specific probabilities for each outcome to compute E(X) and Var(X) accordingly.

Can you interpret the meaning of E(X) = 3.5 in practical terms?

It indicates that over many rolls, the average result tends toward 3.5, even though 3.5 isn't an actual outcome on a single roll.

Additional Resources

Suppose That We Roll A Fair Die And Let X Be The Resulting Number Find E(X), Var(X)

In the realm of probability theory and statistics, understanding the behavior of random variables is fundamental to analyzing uncertainty, risk, and decision-making processes. When considering simple experiments such as rolling a fair die, the random variables involved often serve as foundational examples illustrating core concepts like expectation (mean) and variance. This article provides a comprehensive exploration of the scenario where a fair six-sided die is rolled, and the random variable X represents the outcome. We will delve into the calculation of the expected value $E(X)$, the variance $\operatorname{Var}(X)$, and extend our understanding by examining the properties and implications of these measures in this context.

Understanding the Scenario: Rolling a Fair Die

The Setup of the Experiment

Imagine a standard, six-sided die with faces numbered from 1 to 6. The die is fair, meaning each face has an equal probability of landing face-up after a roll. The experiment involves rolling this die once and recording the resulting number as the random variable X.

- Sample Space (Ω): $\{1, 2, 3, 4, 5, 6\}$
- Probability Distribution: Since the die is fair, each outcome has a probability of $\frac{1}{6}$.

This setup is a classic example in probability theory used to illustrate discrete uniform distributions and their associated statistical measures.

Probability Distribution of X

The probability distribution of the random variable X is the foundation upon which expectation and variance are built.

Discrete Uniform Distribution
Given the fairness of the die:

$$
P(X = k) = \frac{1}{6} \quad \text{for } k = 1, 2, 3, 4, 5, 6.
$$

This distribution is characterized by equal probabilities for all outcomes, making it a discrete uniform distribution.

Expected Value (Mean)
The expected value $E(X)$ is the long-term average of outcomes if the experiment were

repeated many times.

Variance
The variance $\operatorname{Var}(X)$ measures the spread or dispersion of the outcomes around the mean.

Calculating the Expected Value $E(X)$

Definition of Expectation for Discrete Random Variables

The expectation $E(X)$ of a discrete random variable X is defined as:
$$
E(X) = \sum_{k} k \cdot P(X = k).
$$
In our case:
$$
E(X) = \sum_{k=1}^{6} k \cdot P(X = k) = \sum_{k=1}^{6} k \cdot \frac{1}{6}.
$$

Step-by-Step Calculation
$$
E(X) = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6).
$$
The sum of the integers from 1 to 6 is:
$$
1 + 2 + 3 + 4 + 5 + 6 = 21.
$$
Therefore:
$$
E(X) = \frac{1}{6} \times 21 = 3.5.
$$

Interpretation
The expected value of 3.5 indicates the average outcome if the die is rolled many times. Although 3.5 is not a possible outcome on a single roll, it represents the mean over numerous trials, reflecting the central tendency of the uniform distribution.

Calculating the Variance $\operatorname{Var}(X)$

Definition of Variance

The variance of a discrete random variable X is given by:

$$
\operatorname{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2,
$$

where $E[X^2]$ is the expected value of the square of X.

Calculating $E[X^2]$

To find the variance, we first compute $E[X^2]$:

$$
E[X^2] = \sum_{k=1}^{6} k^2 \cdot P(X = k) = \frac{1}{6} \sum_{k=1}^{6} k^2.
$$

Sum of squares from 1 to 6:

$$
1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91.
$$

Thus:

$$
E[X^2] = \frac{1}{6} \times 91 = \frac{91}{6} \approx 15.1667.
$$

Calculating $\operatorname{Var}(X)$

Using the formula:

$$
\operatorname{Var}(X) = E[X^2] - (E[X])^2,
$$

we substitute the values:

$$
\operatorname{Var}(X) = \frac{91}{6} - (3.5)^2 = \frac{91}{6} - 12.25.
$$

Expressing 12.25 as a fraction:

$$
12.25 = \frac{49}{4}.
$$

Express both terms with a common denominator of 12:

$$
\frac{91}{6} = \frac{182}{12}, \quad \frac{49}{4} = \frac{147}{12}.
$$

Hence:

$$
\operatorname{Var}(X) = \frac{182}{12} - \frac{147}{12} = \frac{35}{12} \approx 2.9167.
$$

Interpretation

The variance of $\frac{35}{12}$ indicates the degree of spread in the possible outcomes.

A variance of approximately 2.92 suggests a moderate dispersion around the mean of 3.5, consistent with the uniform distribution's properties.

Deeper Insights and Analytical Perspectives

Properties of the Distribution

The uniform distribution of the die outcomes exhibits several noteworthy properties:

- Symmetry: The distribution is symmetric around the mean, which is 3.5.
- Equal Probabilities: Each outcome has an equal chance, leading to maximum entropy for a distribution over six outcomes with this structure.
- Bounded Variance: The variance is finite and calculable, providing insights into the expected fluctuations around the mean.

Implications for Real-World Applications

While rolling a fair die is a simple theoretical model, its principles extend to more complex random processes:

- Game Theory and Gambling: Understanding expectation and variance helps in assessing game fairness and risk.
- Simulation and Modeling: Discrete uniform distributions serve as baselines for simulating random processes.
- Decision Analysis: Quantifying expected outcomes and variability informs strategic decision-making under uncertainty.

Extensions and Related Concepts

Beyond the basic calculations, several related ideas enrich the understanding of this scenario:

- Higher Moments: Skewness and kurtosis could be analyzed to understand asymmetry and tail behavior.
- Multiple Rolls: Studying the sum or average of multiple independent rolls leads to the Central Limit Theorem's applications.
- Non-Uniform Distributions: Introducing biases or weighted outcomes can model more realistic scenarios, affecting expectation and variance calculations.

Conclusion: The Significance of Expectation and Variance in Discrete Distributions

The exploration of rolling a fair die and analyzing the resulting random variable X exemplifies foundational concepts in probability and statistics. The expectation $E(X) = 3.5$ underscores the average outcome over many trials, while the variance $\operatorname{Var}(X) = \frac{35}{12}$ quantifies the variability inherent in the process. These measures are not only mathematically elegant but also practically valuable, offering insights into the behavior of random systems, guiding strategic decisions, and informing theoretical developments.

By dissecting this simple yet illustrative example, we gain a deeper appreciation for the tools and concepts that underpin the study of randomness. Whether in gaming, modeling, or scientific research, understanding expectation and variance remains crucial for interpreting, predicting, and managing uncertainty in diverse contexts. The principles demonstrated here extend far beyond the roll of a die, serving as cornerstones of probabilistic reasoning and statistical analysis.

Suppose That We Roll A Fair Die And Let X Be The Resulting Number Find E X Var X

Suppose That We Roll A Fair Die And Let X Be The Resulting Number Find E X Var X

Related Articles

- The Internet Was An Avenue Used To Open Up New Markets Around The World. TrueFalse
- The _____ Bones Form The Bridge Of The Nose.A) EthmoidB) LacrimalC) MandibleD) Nasal
- What Level Of Law Enforcement Divides Duties Between Counties And Municipalities?

Back to Home