Three Positive Integers Have A Mean Of 4 And A Range Of 7. Find The Three Numbers.

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Understanding how to find three positive integers based on their mean and range involves applying fundamental concepts of algebra, number properties, and problem-solving strategies. This type of problem appears frequently in math competitions, standardized tests, and classroom exercises, making it a valuable example for students learning how to analyze and solve word problems involving integers.

In this comprehensive article, we will explore the step-by-step process of determining the three positive integers given that their mean is 4 and their range is 7. We'll delve into key mathematical concepts, provide detailed explanations, and include illustrative examples to clarify the solution process. Whether you're a student preparing for exams or someone interested in sharpening your problem-solving skills, this guide will help you understand the underlying principles and improve your ability to tackle similar problems.

Understanding the Problem

Before diving into calculations, it's essential to understand what the problem is asking and what information is provided.

Given:
- The three numbers are positive integers.
- Their mean (average) is 4.
- Their range is 7.

Goals:
- Find the three integers.

Key Concepts Involved

To solve this problem, several mathematical concepts are involved:

1. Mean (Average)

The mean of three numbers \(a\), \(b\), and \(c\) is calculated as:
\[

\text{Mean} = \frac{a + b + c}{3}
\]
Given the mean is 4:
\[
\frac{a + b + c}{3} = 4
\]
which implies:
\[
a + b + c = 12
\]

2. Range

The range of a set of numbers is the difference between the largest and smallest numbers:
\[
\text{Range} = \text{Maximum} - \text{Minimum}
\]
Given the range is 7:
\[
\text{Maximum} - \text{Minimum} = 7
\]

3. Positivity of Integers

All numbers \(a\), \(b\), and \(c\) are positive integers, i.e., greater than 0.

Step-by-Step Solution Approach

To find the three integers, we need to systematically analyze the constraints and possible scenarios.

Step 1: Assign Variables to the Numbers

Let:
- \(a\) = the smallest positive integer,
- \(c\) = the largest positive integer,
- \(b\) = the middle number (which could be either the smallest, middle, or largest depending on the values).

Since the problem involves the range, which is the difference between the maximum and minimum, it's practical to denote:
\[
\text{Minimum} = a, \quad \text{Maximum} = c
\]
and \(b\) as the middle value satisfying:

\[
a \leq b \leq c
\]

Given that the sum of all three is 12:
\[
a + b + c = 12
\]

And the range:
\[
c - a = 7
\]

Step 2: Express the Variables in Terms of One Another

From the range:
\[
c = a + 7
\]

Substitute into the sum:
\[
a + b + (a + 7) = 12
\]
which simplifies to:
\[
2a + b + 7 = 12
\]
\[
b = 12 - 7 - 2a = 5 - 2a
\]

Now, since \(a\), \(b\), and \(c\) are positive integers, and \(a \leq b \leq c\), the following conditions must be satisfied:

- \(a \geq 1\) (positive integers)
- \(b \geq a\)
- \(b \leq c = a + 7\)

Step 3: Determine Valid Values for \(a\)

Because \(b = 5 - 2a\), and \(b\) must be positive:
\[

$$
b \geq 1
$$
$$
5 - 2a \geq 1
$$
$$
-2a \geq -4
$$
$$
2a \leq 4
$$
$$
a \leq 2
$$

Also, $a \geq 1$, so possible values for a are:

- $a = 1$
- $a = 2$

Let's analyze each case separately.

Case 1: $a = 1$

Compute b:
$$
b = 5 - 2(1) = 5 - 2 = 3
$$
Compute c:
$$
c = a + 7 = 1 + 7 = 8
$$

Check if the numbers satisfy the ordering:
$$
a = 1, \quad b = 3, \quad c = 8
$$
Order: $1 \leq 3 \leq 8$ — valid.

Verify the sum:
$$
1 + 3 + 8 = 12
$$
And the range:
$$
8 - 1 = 7
$$

All conditions are met.

Case 2: $a = 2$

Compute b:
$$
b = 5 - 2(2) = 5 - 4 = 1
$$
Compute c:
$$
c = 2 + 7 = 9
$$

Check order:
$$
a = 2, \quad b = 1, \quad c = 9
$$
Order: $1 \leq 2 \leq 9$? No, because $b = 1$ is less than $a = 2$.

But since in the original assignment, a was defined as the smallest number, the order should be $a \leq b \leq c$. Here, $b = 1$ is less than $a=2$, which violates the ordering.

Therefore, this set is invalid unless we reassign the labels, but since the problem specifies the numbers are positive integers with a specific range and mean, and the smallest is a, the set with $a=2$, $b=1$, $c=9$ does not satisfy the ordering assumption.

Conclusion: For the purpose of the problem, the set $(1, 3, 8)$ is valid, and the set with $a=2$ is invalid because it violates the assumption that a is the smallest.

Final Solution and Valid Set of Numbers

Based on the above analysis, the only valid set of three positive integers satisfying the conditions is:

$\boxed{1, 3, 8}$

- Sum check: $1 + 3 + 8 = 12$, which yields a mean of $\frac{12}{3} = 4$.
- Range check: $8 - 1 = 7$.

Thus, the three numbers are 1, 3, and 8.

Additional Insights and Variations

While the specific problem yields a unique set of numbers under the assumptions, similar problems can lead to multiple solutions or require different approaches.

1. Considering Different Orderings

If the problem does not specify the order of the numbers, permutations may be valid as long as they satisfy the sum and range constraints.

2. Extension to Other Averages and Ranges

Changing the mean or range values introduces different equations and solutions. For example:
- If the mean were 5, the sum would be $3 \times 5 = 15$.
- If the range were 5 or 10, similar algebraic steps could be used.

3. General Approach for Similar Problems

- Assign variables to the minimum and maximum.
- Express the middle number in terms of the other variables.
- Use the sum and range constraints to find possible values.
- Verify the ordering and positivity.

Conclusion

In solving the problem of finding three positive integers with a mean of 4 and a range of 7, a systematic approach involving algebraic expressions and logical reasoning is essential. By defining variables, establishing relationships, and applying constraints, the solution reveals that the three numbers are 1, 3, and 8. This problem exemplifies how combining basic properties of numbers with algebraic techniques can lead to clear, logical solutions.

Understanding these methods enhances problem-solving skills and prepares students to tackle a wide range of similar mathematical challenges. Whether for academic purposes or personal interest, mastering such problems deepens comprehension of number properties and algebraic reasoning.

Keywords: positive integers, mean, range, algebra, problem-solving, mathematical reasoning, number properties, equations, sets of numbers

Frequently Asked Questions

What are the three positive integers if their mean is 4 and their range is 7?

The three numbers are 1, 4, and 8.

How do you determine the three numbers given the mean and range constraints?

You set up equations based on the sum (mean × 3) and the difference between the largest and smallest numbers (range), then find all possible positive integer solutions.

Why is it important that the numbers are positive integers in this problem?

Because the problem specifies positive integers, the solutions must be greater than zero, limiting the possible combinations accordingly.

Can the three numbers be the same given the range of 7?

No, they cannot be the same because if all three were equal, the range would be zero, which contradicts the range of 7.

What is the total sum of the three numbers in this problem?

Since the mean is 4, the total sum of the three numbers is 4 × 3 = 12.

How many possible sets of three positive integers satisfy the given conditions?

There are exactly two sets: 1, 4, 8 and 2, 4, 9.

What steps would you follow to find the three numbers in similar problems?

First, determine the total sum from the mean, then use the range to set bounds on the smallest and largest numbers, and finally find all integer combinations that satisfy these conditions.

Additional Resources

Understanding the Problem: Three Positive Integers Have a Mean of 4 and a Range of 7. Find the Three Numbers.

When tackling problems involving three positive integers with specific conditions, such as a given mean and range, it’s essential to break down the problem systematically. This particular problem offers an engaging opportunity to explore the relationships between the sum, individual values, and differences among numbers. The key is to interpret the conditions carefully and then use algebraic reasoning to find the three integers.

Interpreting the Problem Statement

This problem states:

- The three integers are positive integers.
- The mean (average) of these three integers is 4.
- The range of these integers is 7.

From this, we can derive important information:

- Since the mean of three numbers is 4, the sum of these three numbers is:

$$
\text{Sum} = \text{Mean} \times 3 = 4 \times 3 = 12
$$

- The range is the difference between the largest and smallest integers:

$$
\text{Range} = \text{Largest} - \text{Smallest} = 7
$$

The goal is to find the three integers that satisfy these conditions.

Translating Conditions into Mathematical Expressions

Let's denote the three positive integers as:

$$
a, b, c
$$

with the understanding that:

$$
a \leq b \leq c

\]

since the integers are positive and we can order them for clarity.

From the problem, we have:

1. Sum condition:

\[
a + b + c = 12
\]

2. Range condition:

\[
c - a = 7
\]

Given these, our task reduces to finding all positive integers \(a, b, c \) such that:

\[
a + b + c = 12
\]
\[
c - a = 7
\]
\[
a, b, c \in \mathbb{Z}^+ \quad \text{(positive integers)}
\]

Analyzing the Range Condition

The range condition \(c - a = 7 \) indicates that the largest and smallest integers differ by 7.

Since \(a \) and \(c \) are positive integers:

- \(c = a + 7 \)

- \(a \geq 1 \) (since the smallest integer must be positive)

- \(c \geq a + 7 \)

Given that the sum of the three numbers is 12, and \(c = a + 7 \), we have:

\[
a + b + (a + 7) = 12
\]

which simplifies to:

$$
2a + b + 7 = 12
$$

or

$$
b = 12 - 7 - 2a = 5 - 2a
$$

Constraints on the Variables

Since all three are positive integers, the following constraints must hold:

- $a \geq 1$

- $b \geq 1$

- $c = a + 7 \geq 1 + 7 = 8$ (since $a \geq 1$)

From the expression for b:

$$
b = 5 - 2a
$$

To ensure b is positive:

$$
5 - 2a \geq 1
$$
$$
-2a \geq -4
$$
$$
2a \leq 4
$$
$$
a \leq 2
$$

Combined with $a \geq 1$, this gives:

$$
a \in \{1, 2\}

\]

Now, analyze each case:

Case 1: \(a = 1 \)

Calculate \(b \):

\[
b = 5 - 2(1) = 5 - 2 = 3
\]

Calculate \(c \):

\[
c = a + 7 = 1 + 7 = 8
\]

Check the sum:

\[
a + b + c = 1 + 3 + 8 = 12
\]

All are positive integers, and the sum matches. The range:

\[
c - a = 8 - 1 = 7
\]

which matches the given condition.

Case 2: \(a = 2 \)

Calculate \(b \):

\[
b = 5 - 2(2) = 5 - 4 = 1
\]

Calculate \(c \):

\[
c = 2 + 7 = 9
\]

Check the sum:

\[
a + b + c = 2 + 1 + 9 = 12
\]

Again, the sum matches, and the range:

$$
c - a = 9 - 2 = 7
$$

which satisfies the range condition.

Verifying the Solutions

From the above calculations, the potential solutions are:

- Solution 1: $(a, b, c) = (1, 3, 8)$

- Solution 2: $(a, b, c) = (2, 1, 9)$

However, since we initially ordered the integers as $a \leq b \leq c$, the second solution needs to be checked:

$$
a = 2, \quad b = 1, \quad c = 9
$$

Ordering:

$$
1 \leq 2 \leq 9
$$

but the actual a is 2, which is greater than 1, so the order isn't maintained as initially assumed. To comply with the ordered assumption, the solutions should be reordered:

$$
(1, 3, 8)
$$
and

$$
(1, 2, 9)
$$

But note that in the second solution, the set is $\{1, 2, 9\}$, which sum to 12, with a range of $9 - 1 = 8$. This contradicts the range condition of 7. Therefore, the second solution does not satisfy the range condition when ordered properly.

Final Valid Solutions

Given the above, the only valid triplet that satisfies all conditions is:

$$
\boxed{(1, 3, 8)}
$$

which when ordered as $1 \leq 3 \leq 8$, confirms the range:

$$
8 - 1 = 7
$$

and the sum:

$$
1 + 3 + 8 = 12
$$

with the mean:

$$
\frac{12}{3} = 4
$$

Deeper Insights and Alternative Approaches

While the algebraic approach yields the solutions efficiently, it’s insightful to explore alternative methods and reasoning that can help deepen understanding.

Graphical or Systematic Enumeration

- Systematic listing: Since the sum is fixed at 12, and integers are positive, one can list all triplets (a, b, c) with $a \leq b \leq c$ and sum 12, then check the range condition.

- Constraints:
- $a \geq 1$
- $c = a + 7$, so $a \leq c - 7$, and since $c \geq a$, this is consistent.

- Possible triplets:
- $a = 1 \Rightarrow c = 8$, then $b = 12 - a - c = 12 - 1 - 8 = 3$. Sorted: $(1, 3, 8)$.
- $a = 2 \Rightarrow c = 9$, then $b = 12 - 2 - 9 = 1$. Sorted: $(1, 2, 9)$. But range is 9 - 1 = 8, which doesn’t satisfy the range condition of 7. So discard.
- $a = 3 \Rightarrow c = 10$, then $b = 12 - 3 - 10 = -1$, invalid as b must be positive.

Thus, only the first triplet remains valid.

Algebraic Generalization and Constraints

Suppose we define:

$$
a = x, \quad c = x + 7
$$

and

$$
b = 12 - a - c = 12 - x - (x + 7) = 12 - 2x - 7 = 5 - 2x
$$

For a, b, c positive integers:

- $a = x \geq 1$
- $c = x + 7 \geq 8$
- $b = 5 - 2x \geq 1$

From $b \geq 1$:

$$
5 - 2x \geq 1
$$
$$
-2x \geq -4
$$
\

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