Which Equation Represents A Parabola With A Focus Of (4,-3) And A Directrix Of Y = 1?

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Understanding the precise equation of a parabola given its focus and directrix is fundamental in conic sections, especially in algebra and coordinate geometry. In this comprehensive guide, we will explore how to determine the equation of a parabola when the focus is at (4, -3) and the directrix is the line y = 1. We will delve into the geometric principles, derive the relevant formulas, and walk through the step-by-step process to find the accurate equation. This article aims to clarify the concepts for students, educators, and enthusiasts seeking a detailed explanation and practical methodology.

What Is a Parabola and Its Geometric Definition?

A parabola is a symmetric, U-shaped curve that is defined as the set of all points equidistant from a fixed point called the focus and a fixed line called the directrix. This geometric property is crucial in deriving the algebraic equation of the parabola.

Key components:

- Focus: The point inside the parabola that the curve is "centered" around.
- Directrix: A fixed line outside the parabola that helps define the shape.
- Vertex: The point where the parabola changes direction, located midway between the focus and directrix.

Geometric Definition:

> A parabola is the locus of points equidistant from the focus and the directrix.

Understanding this fundamental property allows us to formulate the algebraic equation once the focus and directrix are known.

Understanding the Focus and Directrix in the Given Problem

In our specific case, we have:

- Focus: (4, -3)

- Directrix: $y = 1$

Let's analyze these components:

- The focus is located at $x = 4$, $y = -3$.
- The directrix is a horizontal line $y = 1$.

Since the directrix is horizontal and above the focus, the parabola opens downward. This is consistent with the focus being below the directrix.

Visual Representation:

Imagine plotting the point (4, -3) on the coordinate plane and drawing the line $y = 1$ above it. The parabola opens downward, with its vertex somewhere between the focus and the directrix.

Determining the Parabola’s Orientation and Key Parameters

Before deriving the equation, let's analyze the key parameters:

1. Distance Between Focus and Directrix

The distance between the focus and the directrix line gives us the value of 2a, where a is a key parameter in the parabola's standard form.

- Distance: $|y_{focus} - y_{directrix}| = |(-3) - 1| = 4$

2. Location of the Vertex

The vertex lies midway between the focus and the directrix along the axis of symmetry.

- Since the directrix is $y = 1$ and the focus is at $y = -3$, the vertex's y-coordinate is:

$y_{vertex} = (y_{focus} + y_{directrix}) / 2 = (-3 + 1) / 2 = -1$

- The x-coordinate of the vertex is the same as that of the focus (since the parabola opens vertically):

$x_{vertex} = 4$

Thus, the vertex is at (4, -1).

3. Axis of Symmetry

The parabola's axis of symmetry is vertical, passing through the focus and vertex, at $x = 4$.

4. Distance from Vertex to Focus (or Directrix)

This distance is:

- $|y_{focus} - y_{vertex}| = |-3 - (-1)| = 2$

This equals $|a|$ in the parabola's standard form.

Deriving the Equation of the Parabola

Given the above information, we can now proceed to derive the parabola's equation.

1. Recognize the Standard Form

For a parabola opening vertically, the standard form is:

$$ (x - h)^2 = 4a(y - k) $$

where:

- (h, k) is the vertex,
- a determines the distance from the vertex to the focus and directrix,
- the parabola opens upward if $a > 0$, downward if $a < 0$.

2. Identify the Vertex and Parameter a

From earlier calculations:

- Vertex: $(h, k) = (4, -1)$
- Distance from vertex to focus: $|a| = 2$

Since the focus is below the vertex (focus at $y = -3$, vertex at $y = -1$), the parabola opens downward, so $a < 0$.

Therefore:

$$ a = -2 $$

3. Write the Equation

Plugging into the standard form:

$$ (x - 4)^2 = 4 \times (-2) \times (y + 1) $$

Simplify:

$$ (x - 4)^2 = -8(y + 1) $$

This is the equation of the parabola in standard form.

Alternative Form and Final Equation

The derived equation:

$$ (x - 4)^2 = -8(y + 1) $$

fully describes the parabola with focus at (4, -3) and directrix y = 1.

Graphical interpretation:

- The parabola opens downward.
- It is symmetric about the line x = 4.
- The vertex at (4, -1) is equidistant from the focus and directrix.

Verification of the Equation

To ensure correctness, verify that:

1. The focus (4, -3) satisfies the equation.

Substitute x = 4, y = -3:

$$ (4 - 4)^2 = -8(-3 + 1) $$
$$ 0 = -8(-2) $$
$$ 0 = 16 $$

Contradiction? Yes, but note that the focus is not on the parabola itself but on the set of points satisfying the parabola's definition. The focus should satisfy the property that any point P on the parabola is equidistant from the focus and the directrix.

2. Distance from focus to a point on the parabola

Pick a point on the parabola, say the vertex (4, -1), and verify the distances:

- Distance from (4, -1) to focus (4, -3):

$$ \sqrt{(4 - 4)^2 + (-1 + 3)^2} = \sqrt{0 + 4} = 2 $$

- Distance from (4, -1) to directrix y=1:

Vertical distance:

$$ | -1 - 1 | = 2 $$

Equal distances confirm the vertex's position.

Conclusion: The equation is correct and consistent with the geometric properties.

Summary of the Equation and Its Significance

The final and most accurate algebraic representation of the parabola with focus at (4, -3) and directrix y = 1 is:

$$ \boxed{(x - 4)^2 = -8(y + 1)} $$

This equation can be used for graphing, analysis, and solving related problems in algebra and coordinate geometry.

Additional Notes and Variations

- Horizontal Parabolas: If the focus and directrix were oriented differently (e.g., focus at (x, y) and directrix vertical), the standard form would change accordingly.
- Vertex Form: The derived form is already in vertex form, emphasizing the importance of vertex and focus placement.
- Applications: Parabolas are used in physics (reflectors), engineering (antennae), and mathematics for modeling various phenomena.

Conclusion

Identifying the equation of a parabola from its focus and directrix involves understanding the fundamental geometric property of the parabola, calculating key points like the vertex, and applying the standard parabola equation. For the focus at (4, -3) and directrix y=1, the parabola opens downward with the equation:

$$ (x - 4)^2 = -8(y + 1) $$

This detailed process demonstrates the power of conic section principles in algebraic problem-solving and provides a clear pathway for students and professionals to analyze similar problems in geometry.

Keywords: parabola equation, focus, directrix, vertex, standard form, conic sections, coordinate geometry, algebra, parabola derivation, focus-directrix property

Frequently Asked Questions

What is the general approach to find the equation of a parabola given its focus and directrix?

To find the equation, use the definition of a parabola as the set of points equidistant from the focus and the directrix. Set up the distance formula for a generic point (x, y) to the focus and to the directrix, then equate and simplify to get the parabola's equation.

Given the focus at (4, -3) and the directrix y = 1, how do you determine the vertex of the parabola?

The vertex lies midway between the focus and the directrix along the axis of symmetry. Since the directrix is y=1 and the focus is at y=-3, the vertex's y-coordinate is at (1 + (-3))/2 = -1, and the x-coordinate is the same as the focus, x=4. So, the vertex is at (4, -1).

What is the length of the focal distance (p) for this parabola?

The focal distance p is the distance from the vertex to the focus (or the directrix). The focus is at (4, -3) and the vertex at (4, -1), so p = | -3 - (-1) | = 2.

How do you write the equation of a parabola with a vertical axis of symmetry, focus at (4, -3), and vertex at (4, -1)?

Since the parabola opens downward (focus below the vertex), the equation is (x - h)^2 = 4p(y - k), where (h, k) is the vertex. Here, h=4, k=-1, and p=-2, so the equation is (x - 4)^2 = 4(-2)(y + 1), simplifying to (x - 4)^2 = -8(y + 1).

What is the final simplified equation of the parabola with focus at (4, -3) and directrix y=1?

The equation is (x - 4)^2 = -8(y + 1).

Is the parabola opening upward or downward in this case?

The parabola opens downward because the focus is below the vertex, and the coefficient in the equation (x - 4)^2 = -8(y + 1) is negative, indicating a downward opening.

How does changing the focus or directrix affect the parabola's shape and equation?

Moving the focus or directrix shifts the parabola's position and alters the vertex and focal length, changing the shape and size. The vertex moves accordingly, and the equation's parameters adjust to reflect these changes.

Can this parabola's equation be written in a different form?

Yes, the equation can be expanded or rearranged, but the vertex form (x - h)^2 = 4p(y - k) is most straightforward for identifying the parabola's key features. For example, expanding yields x^2 - 8x + 16 = -8(y + 1), which can be rearranged as needed.

How do you verify that the equation correctly represents the parabola with the given focus and directrix?

Substitute the focus point into the equation and check if the distance from the point to the focus equals the distance to the directrix. Alternatively, verify that the focus satisfies the parabola's geometric definition based on the derived equation.

Additional Resources

Understanding Parabolas and Their Equations: A Deep Dive into Focus-Directrix Formulation

When exploring conic sections, parabolas hold a unique position due to their distinctive geometric and algebraic properties. One fundamental question often posed in conic section studies is: Which equation represents a parabola with a given focus and directrix? Specifically, understanding how to derive the equation of a parabola when its focus and directrix are known is essential for students, educators, and enthusiasts alike. In this review, we will analyze the problem: "Which equation represents a parabola with a focus of (4, -3) and a directrix of y = 1?" and explore the comprehensive methodology to arrive at the accurate equation.

Fundamentals of Parabola Geometry

What Is a Parabola?

A parabola is a symmetric, U-shaped curve that can be defined as the set of all points equidistant from a fixed point called the focus and a fixed line called the directrix. This geometric property is central to deriving the algebraic equation of the parabola.

Key Elements:
- **Focus:** The point inside the parabola that the curve "opens" towards.
- **Directrix:** A fixed line outside the parabola that helps define the parabola’s shape.
- **Vertex:** The point where the parabola reaches its minimum or maximum, located midway between the focus and directrix.
- **Axis of symmetry:** A line passing through the focus and perpendicular to the directrix, dividing the parabola into mirror images.

Standard Parabola Equation (Vertex Form)

Depending on the parabola’s orientation (opening upward, downward, left, or right), the standard forms are:
- Vertical parabola (opening up/down): $(x - h)^2 = 4p(y - k)$
- Horizontal parabola (opening left/right): $(y - k)^2 = 4p(x - h)$

Here, (h, k) is the vertex, and p is the distance from the vertex to the focus (or equivalently, from the vertex to the directrix).

Focus-Directrix Definition and Its Algebraic Implications

Geometric Perspective

The defining property of a parabola: any point P on the parabola is equidistant from the focus F and the directrix D. Mathematically:

$$
\text{Distance}(P, F) = \text{Distance}(P, \text{directrix})
$$

Given:
- Focus $F = (4, -3)$
- Directrix $y = 1$

The goal is to find the equation of the parabola satisfying this property.

Step-by-Step Approach

1. Identify the orientation of the parabola:
- Since the directrix is $y = 1$, which is horizontal, and the focus is at $(4, -3)$, which is below the directrix, the parabola opens downward (toward the focus).

2. Determine the vertex:
- The vertex V lies midway between the focus and the directrix along the axis of symmetry.
- The directrix is at $y = 1$, and the focus is at $y = -3$.
- The midpoint along the y-axis:

$$
y_v = \frac{-3 + 1}{2} = -1
$$

- The x-coordinate of the vertex is the same as the focus, since the parabola opens vertically:

$$
x_v = 4
$$

- Therefore, the vertex is at $(4, -1)$.

3. Determine the focal length p:
- p is the distance from the vertex to the focus:

$$
p = | y_{focus} - y_{vertex} | = | -3 - (-1) | = 2
$$

- Since the parabola opens downward (toward the focus), p is negative in the standard form:

$$
p = -2
$$

4. Write the parabola’s equation:
- For a vertical parabola with vertex (h, k), the standard form is:

$$
(x - h)^2 = 4p(y - k)
$$

- Plugging in $h = 4$, $k = -1$, and $p = -2$:

$$
(x - 4)^2 = 4 \times (-2) \times (y + 1)
$$

$$
(x - 4)^2 = -8(y + 1)
$$

Deriving the Equation: Step-by-Step Breakdown

Step 1: Confirm the Geometric Setup

- The focus is at $(4, -3)$.
- The directrix is at $y = 1$.
- The parabola opens downward because the focus is below the directrix.

Step 2: Find the Vertex Coordinates

- Since the parabola is symmetric about the axis passing through the focus and perpendicular to the directrix, the vertex is midway along this axis.
- The focus’s y-coordinate: -3.
- The directrix’s y-coordinate: 1.
- Midpoint (vertex y-coordinate): $\frac{-3 + 1}{2} = -1$.
- The x-coordinate of the vertex remains the same as the focus: 4.

Vertex: $(4, -1)$.

Step 3: Calculate p

- Distance from vertex to focus:

$$
p = y_{focus} - y_{vertex} = -3 - (-1) = -2
$$

- The negative sign indicates the parabola opens downward.

Step 4: Write the Equation

- Using the standard form for a vertical parabola:
$$
(x - h)^2 = 4p(y - k)
$$
- Substitute $h = 4$, $k = -1$, $p = -2$:
$$
(x - 4)^2 = 4 \times (-2) \times (y + 1)
$$
$$
(x - 4)^2 = -8(y + 1)
$$

This is the algebraic equation representing the parabola with the given focus and directrix.

Verifying the Equation

Check that the focus lies on the parabola

- The focus at $(4, -3)$ should satisfy the equation:
$$
(4 - 4)^2 = -8(-3 + 1)
$$
$$
0 = -8 \times (-2) = 16
$$

- The left side is 0, but the right side is 16. So, the focus does not satisfy the equation directly because the focus is outside the parabola’s locus of points described by the equation.

Note: This discrepancy arises because the standard form's equation describes the parabola's locus, but the focus point itself is not necessarily on the parabola unless substituted into the equation of the parabola’s point set. To verify, we check whether the focus point maintains the property of equal distances from the focus and directrix:

- Distance from focus $(4, -3)$ to point (x, y):
$$
\sqrt{(x - 4)^2 + (y + 3)^2}
$$
- Distance from point (x, y) to directrix $y = 1$:
$$
| y - 1 |
$$

Since the focus point is on the parabola, when $(x, y) = (4, -3)$:

- Distance to focus:
$$
\sqrt{(4 - 4)^2 + (-3 + 3)^2} = 0
$$
- Distance to directrix:
$$
|-3 - 1| = 4
$$

They're not equal, but this indicates the focus itself isn't on the parabola (which is expected). The parabola's locus is the set of points equidistant from the focus and directrix, not necessarily passing through the focus point itself.

Alternative Formulations and Generalizations

Standard Equation in Vertex Form

The derived equation:
$$
(x - 4)^2 = -8(y + 1)
$$
is in vertex form and explicitly shows the parabola’s opening downward with vertex at $(4, -1)$.

Conic Section Form

Alternatively, the parabola can be expressed in the general quadratic form:
$$
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0
$$
but

[Which Equation Represents A Parabola With A Focus Of 4 3]

And A Directrix Of Y 1

Which Equation Represents A Parabola With A Focus Of 4 3 And A Directrix Of Y 1

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