Write The Equation In Spherical Coordinates. (a) 3x2 2x 3y2 3z2 = 0 (b) 5x 4y 5z = 1

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Introduction

When working with three-dimensional equations, especially those involving spheres, cones, or other curved surfaces, Cartesian coordinates often become cumbersome. Spherical coordinates provide a powerful alternative, simplifying the representation of many geometric objects and equations. This coordinate system expresses points in space using a radius and two angles, making it particularly advantageous in fields like physics, engineering, and mathematics.

In this article, we delve into transforming two specific equations from Cartesian to spherical coordinates:

- Equation (a): $3x^2 + 2x + 3y^2 + 3z^2 = 0$
- Equation (b): $5x + 4y + 5z = 1$

Understanding how to convert these equations into spherical coordinates not only aids in visualizing the surfaces they represent but also enhances problem-solving efficiency in three-dimensional analysis.

Fundamentals of Spherical Coordinates

Before converting specific equations, it's essential to grasp the basics of spherical coordinates.

Definition and Conversion Relations

Spherical coordinates (r, θ, ϕ) relate to Cartesian coordinates (x, y, z) as follows:

- $x = r \sin \phi \cos \theta$
- $y = r \sin \phi \sin \theta$
- $z = r \cos \phi$

where:

- $r \geq 0$: The distance from the origin to the point.
- $\theta \in [0, 2\pi)$: The azimuthal angle in the xy-plane from the positive x-axis.
- $\phi \in [0, \pi]$: The polar angle from the positive z-axis downward.

Key Relationships

- $x^2 + y^2 + z^2 = r^2$
- $\sin \phi = \frac{\sqrt{x^2 + y^2}}{r}$
- $\cos \phi = \frac{z}{r}$

These relationships facilitate converting Cartesian equations into spherical form.

Transforming Equation (a): $3x^2 + 2x + 3y^2 + 3z^2 = 0$

Step 1: Rewrite the Equation

The original equation:

$$
3x^2 + 2x + 3y^2 + 3z^2 = 0
$$

Group similar terms:

$$
3x^2 + 3y^2 + 3z^2 + 2x = 0
$$

Factor out 3 from the quadratic terms:

$$
3(x^2 + y^2 + z^2) + 2x = 0

$$
\]
$$

Since $x^2 + y^2 + z^2 = r^2$, we can rewrite:

$$
3r^2 + 2x = 0
$$

Expressing x in spherical coordinates:

$$
x = r \sin \phi \cos \theta
$$

Substitute into the equation:

$$
3r^2 + 2r \sin \phi \cos \theta = 0
$$

Step 2: Simplify the Equation

Divide both sides by r (considering $r \neq 0$):

$$
3r + 2 \sin \phi \cos \theta = 0
$$

Solve for r:

$$
r = - \frac{2}{3} \sin \phi \cos \theta
$$

Step 3: Interpret the Result

The spherical coordinate form of equation (a):

$$
r = - \frac{2}{3} \sin \phi \cos \theta
$$

This describes a surface where the radius r depends on the angles

ϕ and θ. The negative sign indicates the surface extends into the negative r direction, which in spherical coordinates can be interpreted as pointing in the opposite direction of $\hat{r}$.

Note: Since $r \geq 0$ in standard spherical coordinates, the negative value suggests the surface is symmetric with respect to the origin and can be re-expressed to incorporate the sign:

$$
r = \left| \frac{2}{3} \sin \phi \cos \theta \right|
$$

or explicitly noting the directionality in the context of the surface.

Summary of Equation (a):

> The surface described by $3x^2 + 2x + 3y^2 + 3z^2 = 0$ in spherical coordinates is given by:

$$
r = - \frac{2}{3} \sin \phi \cos \theta
$$

indicating a surface symmetric about the origin, with the radius depending on the angles.

Transforming Equation (b): $5x + 4y + 5z = 1$

Step 1: Substitute Cartesian to Spherical Coordinates

Express x, y, z:

- $x = r \sin \phi \cos \theta$
- $y = r \sin \phi \sin \theta$
- $z = r \cos \phi$

Plug into the linear equation:

$$
5(r \sin \phi \cos \theta) + 4(r \sin \phi \sin \theta) + 5(r \cos \phi) = 1
$$

Factor out r:

$$
r [5 \sin \phi \cos \theta + 4 \sin \phi \sin \theta + 5 \cos \phi] = 1
$$

Step 2: Simplify the Expression

The entire expression in brackets is:

$$
5 \sin \phi \cos \theta + 4 \sin \phi \sin \theta + 5 \cos \phi
$$

This can be viewed as a linear combination of sines and cosines. To simplify, recognize that:

$$
A \cos \theta + B \sin \theta = R \cos (\theta - \alpha)
$$

where:

- $R = \sqrt{A^2 + B^2}$
- $\alpha = \arctan \left(\frac{B}{A} \right)$

Identify:

$$
A = 5 \sin \phi, \quad B = 4 \sin \phi
$$

Calculate R:

$$
R = \sqrt{(5 \sin \phi)^2 + (4 \sin \phi)^2} = \sin \phi \sqrt{25 + 16} = \sin \phi \sqrt{41}
$$

Find α:

$$
\alpha = \arctan \left(\frac{4 \sin \phi}{5 \sin \phi} \right) = \arctan \left(\frac{4}{5} \right)
$$

which is a constant angle:

$$
\alpha = \arctan \left(\frac{4}{5} \right) \approx 38.66^\circ
$$

Now, rewrite the bracket:

$$
A \cos \theta + B \sin \theta = R \cos (\theta - \alpha)
$$

So the entire expression becomes:

$$
r [\sin \phi \sqrt{41} \cos (\theta - \alpha) + 5 \cos \phi] = 1
$$

Therefore:

$$
r \left(\sin \phi \sqrt{41} \cos (\theta - \alpha) + 5 \cos \phi \right) = 1
$$

Step 3: Express r explicitly

Solve for r:

$$
r = \frac{1}{ \sin \phi \sqrt{41} \cos (\theta - \alpha) + 5 \cos \phi }
$$

This is the spherical coordinate form of the plane equation.

Note: The denominator must not be zero; otherwise, the surface is not defined at those points.

Summary of Both Equations in Spherical Coordinates

Equation	Spherical Coordinates Representation	Remarks
(a) $3x^2 + 2x + 3y^2 + 3z^2 = 0$	$r = - \frac{2}{3} \sin \phi \cos \theta$	Represents a surface symmetric about the origin; the negative

indicates direction. |
| (b) $5x + 4y + 5z = 1$ | $r = \frac{1}{ \sin \phi \sqrt{41} \cos (\theta - \alpha) + 5 \cos \phi }$ | Describes

Frequently Asked Questions

How do you convert the equation $3x^2 + 2x + 3y^2 + 3z^2 = 0$ into spherical coordinates?

First, recall the transformations: $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$. Substitute these into the equation: $3(r \sin\theta \cos\varphi)^2 + 2(r \sin\theta \cos\varphi) + 3(r \sin\theta \sin\varphi)^2 + 3(r \cos\theta)^2 = 0$. Simplify to express the equation entirely in terms of r, θ, and φ.

What is the general approach to write Cartesian equations like $5x + 4y + 5z = 1$ in spherical coordinates?

Replace x, y, and z with their spherical coordinate equivalents: $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$. Then, the equation becomes $5(r \sin\theta \cos\varphi) + 4(r \sin\theta \sin\varphi) + 5(r \cos\theta) = 1$. Simplify to express r in terms of θ and φ.

Are there specific challenges in converting quadratic equations like $3x^2 + 3y^2 + 3z^2 = 0$ into spherical coordinates?

Yes. Quadratic equations involving sums of squares often simplify to expressions involving r^2, since $x^2 + y^2 + z^2 = r^2$. In this case, the equation simplifies to $3r^2 = 0$, indicating a point at the origin. The challenge is recognizing and simplifying such sums correctly during the conversion.

Can all Cartesian equations be directly converted into spherical coordinates, and what are the limitations?

Most equations involving x, y, and z can be converted into spherical coordinates by substitution. However, equations involving inequalities or piecewise definitions may become more complex. Additionally, some equations may simplify to constant or trivial solutions after substitution, indicating specific geometric entities like points or spheres.

How do you interpret the equation $5x + 4y + 5z = 1$

in spherical coordinates geometrically?

This is a plane in Cartesian coordinates. When converted to spherical coordinates, it represents the set of points (r, θ, φ) satisfying the linear equation. Geometrically, it describes the intersection of the plane with the sphere of radius r, which can be visualized as a conical surface or a planar slice depending on the values of r, θ, and φ.

Additional Resources

Write The Equation In Spherical Coordinates: An In-Depth Investigation

In the realm of mathematical analysis and applied physics, the translation of Cartesian equations into spherical coordinates is a fundamental task that reveals the geometric and physical significance of a surface or volume. This article aims to rigorously explore the process of rewriting two distinct equations—namely, $3x^2 + 2x + 3y^2 + 3z^2 = 0$ and $5x + 4y + 5z = 1$—in spherical coordinates. We will analyze these equations in detail, elucidate their geometric interpretations, and discuss their implications in various scientific contexts.

Understanding Spherical Coordinates

Before delving into the specific equations, it is essential to review the fundamentals of spherical coordinates.

Definitions and Transformations

The spherical coordinate system (ρ, θ, ϕ) relates to Cartesian coordinates (x, y, z) through the following relations:

- $x = \rho \sin \phi \cos \theta$
- $y = \rho \sin \phi \sin \theta$
- $z = \rho \cos \phi$

Where:

- $\rho \geq 0$ is the radial distance from the origin.
- $0 \leq \phi \leq \pi$ is the polar angle measured from the positive z-axis.
- $0 \leq \theta < 2\pi$ is the azimuthal angle in the $x y$-plane from the positive x-axis.

The relations allow us to convert Cartesian equations into spherical form, revealing their geometric nature more transparently.

Equation (a): $3x^2 + 2x + 3y^2 + 3z^2 = 0$

This quadratic form appears straightforward but warrants careful analysis to understand its geometric implications after conversion.

Step 1: Simplify the Cartesian Equation

Rewrite the equation:

$$
3x^2 + 2x + 3y^2 + 3z^2 = 0
$$

Group similar terms:

$$
3x^2 + 3y^2 + 3z^2 + 2x = 0
$$

Divide through by 3:

$$
x^2 + y^2 + z^2 + \frac{2}{3}x = 0
$$

Step 2: Complete the Square

Express as:

$$
x^2 + \frac{2}{3}x + y^2 + z^2 = 0
$$

Complete the square for the x-terms:

$$
x^2 + \frac{2}{3}x = \left(x + \frac{1}{3}\right)^2 - \frac{1}{9}
$$

Substitute back:

$$
\left(x + \frac{1}{3}\right)^2 - \frac{1}{9} + y^2 + z^2 = 0
$$

Rearranged:

$$
\left(x + \frac{1}{3}\right)^2 + y^2 + z^2 = \frac{1}{9}
$$

This is a standard sphere equation centered at $\left(-\frac{1}{3}, 0, 0\right)$ with radius $\frac{1}{3}$.

Step 3: Convert to Spherical Coordinates

Recall the coordinate transformations:

$$
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi
$$

Express the shifted x-coordinate:

$$
x + \frac{1}{3} = \rho \sin \phi \cos \theta + \frac{1}{3}
$$

Substitute into the sphere equation:

$$
\left(\rho \sin \phi \cos \theta + \frac{1}{3}\right)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi)^2 = \frac{1}{9}
$$

Expand and simplify:

$$
\rho^2 \sin^2 \phi \cos^2 \theta + \frac{2}{3}\rho \sin \phi \cos \theta + \frac{1}{9} + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi = \frac{1}{9}
$$

Combine terms:

$$
\rho^2 (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi) + \frac{2}{3} \rho \sin \phi \cos \theta + \frac{1}{9} = \frac{1}{9}
$$

Since $\cos^2 \theta + \sin^2 \theta = 1$:

$$
\rho^2 (\sin^2 \phi (1) + \cos^2 \phi) + \frac{2}{3} \rho \sin \phi \cos \theta + \frac{1}{9} = \frac{1}{9}
$$

Note that $\sin^2 \phi + \cos^2 \phi = 1$, so:

$$
\rho^2 (1) + \frac{2}{3} \rho \sin \phi \cos \theta + \frac{1}{9} = \frac{1}{9}
$$

Subtract $\frac{1}{9}$ from both sides:

$$
\rho^2 + \frac{2}{3} \rho \sin \phi \cos \theta = 0
$$

Step 4: Interpret the Spherical Equation

The resulting spherical form:

$$
\rho^2 + \frac{2}{3} \rho \sin \phi \cos \theta = 0
$$

Factor ρ:

$$
\rho \left(\rho + \frac{2}{3} \sin \phi \cos \theta\right) = 0
$$

Thus, the solutions are:

1. $\rho = 0$: The origin point.
2. $\rho = -\frac{2}{3} \sin \phi \cos \theta$: A surface where the radial distance depends on angles.

Since $\rho \geq 0$, the second solution is valid only when:

$$
-\frac{2}{3} \sin \phi \cos \theta \geq 0
$$

This inequality constrains the angular domain for the surface.

Geometric Interpretation:

The original equation describes a sphere centered at \(\left(-\frac{1}{3}, 0, 0\right)\) with radius \(\frac{1}{3}\). Converting to spherical coordinates reveals a surface where the radial distance \(\rho\) depends on the angles \(\phi\) and \(\theta\), with a sign constraint ensuring physical relevance (\(\rho \geq 0\)). This transformation emphasizes the symmetry of the sphere and how it appears from the origin in spherical coordinates.

Equation (b): \(5x + 4y + 5z = 1 \)

This linear equation defines a plane in Cartesian space. Its conversion to spherical coordinates provides insight into the plane's orientation relative to the origin.

Step 1: Express in Spherical Coordinates

Substitute the transformations:

\[
x = \rho \sin \phi \cos \theta
\]
\[
y = \rho \sin \phi \sin \theta
\]
\[
z = \rho \cos \phi
\]

Plug into the plane equation:

\[
5 (\rho \sin \phi \cos \theta) + 4 (\rho \sin \phi \sin \theta) + 5 (\rho \cos \phi) = 1
\]

Factor \(\rho\):

\[
\rho \left(5 \sin \phi \cos \theta + 4 \sin \phi \sin \theta + 5 \cos \phi \right) = 1
\]

Step 2: Isolate \(\rho\)

$$
\rho = \frac{1}{5 \sin \phi \cos \theta + 4 \sin \phi \sin \theta + 5 \cos \phi}
$$

This is the spherical form of the plane, expressing the radial distance ρ as a function of the angles ϕ and θ.

Step 3: Geometric Implications

The denominator:

$$
D(\phi, \theta) = 5 \sin \phi \cos \theta + 4 \sin \phi \sin \theta + 5 \cos \phi
$$

represents the projection of the point (x, y, z) onto the plane's normal vector $\mathbf{n} = (5, 4, 5)$.

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