Can A Cubic Function With Real Coefficients Have Two Real Zerosand One Complex Zero? Explain.

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Understanding the nature of polynomial zeros is fundamental in algebra and calculus. Specifically, the question of whether a cubic function with real coefficients can have exactly two real zeros and one complex zero is an intriguing one. The answer hinges on the properties of polynomials with real coefficients and how their zeros are distributed in the complex plane. This article explores this question in depth, examining the fundamental concepts of polynomial roots, the Fundamental Theorem of Algebra, conjugate pairs, and the implications for cubic functions.

Fundamental Concepts of Polynomial Zeros

Zeros of a Polynomial

A zero (or root) of a polynomial function \(f(x)\) is a value \(x = r\) such that \(f(r) = 0\). The zeros can be real or complex numbers. The number of zeros, counting multiplicities, for an nth-degree polynomial is exactly \(n\).

Degree and Zeros

- For a cubic polynomial (\(degree = 3\)), there are exactly three zeros, counting multiplicities.
- These zeros can be real or complex, but their nature is governed by certain algebraic rules, especially when coefficients are real.

The Fundamental Theorem of Algebra and Zero Distribution

Statement of the Theorem

The Fundamental Theorem of Algebra states that every non-constant polynomial function with complex coefficients has at least one complex root. For polynomials with real coefficients, this theorem implies that all roots are either real or occur in conjugate pairs.

Implication for Real Coefficient Polynomials

- If the polynomial has real coefficients, then non-real roots must come in conjugate pairs.
- That is, if $a + bi$ (with $b \neq 0$) is a root, then its conjugate $a - bi$ is also a root.

Can a Cubic Polynomial with Real Coefficients Have Two Real Zeros and One Complex Zero?

Analyzing the Possibility

Given the fundamental properties, let's consider the possible configurations of roots for a cubic polynomial $f(x)$ with real coefficients:
- All three roots are real.
- One real root and two complex conjugate roots.
- A repeated root scenario (multiplicity greater than one), which can involve real roots or complex roots.

The key question is whether the last configuration—exactly two real roots and one complex root—is possible.

Understanding the Root Structure

Since complex roots of polynomials with real coefficients always come in conjugate pairs, the only way to have a single complex zero is if it is part of a conjugate pair. Therefore:
- Any non-real root must be paired with its conjugate.
- This pairing ensures roots occur in conjugate pairs, which are both non-real or both real.

Given this, the possible root structure for a cubic polynomial with real coefficients is:
- Three real zeros (all roots are real).
- One real zero and a pair of complex conjugate zeros.

But wait! This suggests that the second scenario is indeed possible, where:
- The polynomial has one real zero and two complex zeros, which are conjugates of each other.

Detailed Explanation of the Roots of a Cubic with Real Coefficients

Case 1: Three Real Zeros

- The cubic polynomial's graph intersects the x-axis at three points.
- The zeros can be distinct or repeated.
- The zeros may be irrational or rational, but all are real numbers.

Case 2: One Real Zero and Two Complex Zeros

- The polynomial has exactly one real zero.
- The other two roots are complex conjugates of each other.
- This scenario occurs when the polynomial's discriminant is negative.

Role of the Discriminant in Determining Zero Nature

Discriminant of a Cubic Polynomial

The discriminant (Δ) of a cubic polynomial $ax^3 + bx^2 + cx + d$ provides critical information about the nature and multiplicity of its roots:
- $\Delta > 0$: Three distinct real roots.
- $\Delta = 0$: Multiple roots, at least two roots coincide.
- $\Delta < 0$: One real root and two complex conjugate roots.

Implication for Our Question

- When $\Delta < 0$, the cubic has exactly one real zero and two complex conjugate zeros.
- When $\Delta > 0$, all roots are real.
- When $\Delta = 0$, roots are real with multiplicities, possibly leading to repeated roots.

Thus, it is entirely possible for a cubic polynomial with real coefficients to have:
- Exactly two real zeros and one complex zero (which is part of a conjugate pair), provided the complex zeros are conjugates and the real zeros are real roots.

Clarifying the Terminology: "Two Real Zeros and One Complex Zero"

In common language, "zeros" refer to roots of the polynomial. When the question states "two real zeros and one complex zero," it might be ambiguous whether:
- The "complex zero" is considered one root (which would be a single complex number), or
- The two complex zeros refer to a conjugate pair (which is standard), and the "one real zero" is the real root.

Correct interpretation:
- The polynomial has one real zero and a pair of complex conjugate zeros.
- Since complex zeros occur in conjugate pairs for polynomials with real coefficients, the total number of roots is three: one real and two complex conjugates.

Summary:
- Yes, a cubic polynomial with real coefficients can have two complex zeros (which are conjugates) and one real zero.
- The total count of zeros (including multiplicity) is three.

- The roots' nature depends on the discriminant:
- Negative discriminant: one real and two complex conjugates.
- Positive discriminant: three real roots.
- Zero discriminant: at least two roots coincide.

Conclusion and Final Remarks

The question of whether a cubic function with real coefficients can have two real zeros and one complex zero is rooted in the fundamental properties of polynomial roots and the conjugate root theorem. The answer is affirmative: a cubic polynomial with real coefficients can indeed have exactly one real zero and a pair of complex conjugate zeros. This scenario is common and well-understood within algebraic theory, especially when the discriminant is negative, indicating the presence of complex conjugate roots.

Understanding this root structure is vital for analyzing polynomial behavior, graphing functions, and solving equations in algebra, calculus, and applied mathematics. Recognizing that complex roots always come in pairs for polynomials with real coefficients helps clarify the distribution of zeros and ensures a comprehensive grasp of polynomial root behavior.

In summary:
- A cubic function with real coefficients can have two complex zeros (conjugate pair) and one real zero.
- The key condition is the discriminant being negative.
- This root configuration is consistent with the fundamental theorems governing polynomial roots and their behavior over the real and complex planes.

Frequently Asked Questions

Can a cubic function with real coefficients have exactly two real zeros and one complex zero?

No, a cubic function with real coefficients cannot have exactly two real zeros and one complex zero because complex zeros of polynomials with real coefficients always come in conjugate pairs. Therefore, if there is a complex zero, its conjugate must also be a zero, resulting in either three real zeros or one real zero and two complex conjugate zeros.

Why do complex zeros of polynomials with real coefficients always come in conjugate pairs?

Because when a polynomial has real coefficients, the conjugate of any complex zero must also be a zero to ensure that the polynomial's coefficients remain real after factoring. This conjugate symmetry ensures that non-real roots appear in pairs.

What are the possible zero configurations for a cubic polynomial with real coefficients?

A cubic polynomial with real coefficients can have three real zeros or one real zero and two complex conjugate zeros. It cannot have exactly two real zeros and one complex zero without its conjugate.

If a cubic polynomial with real coefficients has one real zero, what can be said about its other zeros?

If it has one real zero, the other two zeros could be either both real (if the quadratic factor has real roots) or a pair of complex conjugate zeros (if the quadratic factor has complex roots). The overall possibilities are either three real zeros or one real and two complex conjugates.

Is it possible for a cubic polynomial with real coefficients to have a zero that is purely complex (not real)?

A zero of a polynomial with real coefficients is either real or part of a complex conjugate pair. A zero cannot be purely complex without its conjugate also being a zero, ensuring the polynomial's coefficients remain real.

What implications does the conjugate pair rule have for the zeros of cubic functions with real coefficients?

The conjugate pair rule implies that non-real zeros of cubic functions with real coefficients must come in pairs. Therefore, any complex zeros always appear as conjugate pairs, meaning the total number of real zeros can be one or three, but not two alone.

Additional Resources

Can A Cubic Function With Real Coefficients Have Two Real Zeros and One Complex Zero?

The question posed—Can a cubic function with real coefficients have two real zeros and one complex zero?—delves into the fundamental properties of polynomial functions, their roots, and the implications of the coefficients being real. This inquiry not only touches on algebraic theory but also explores the nature of roots in polynomial equations, their multiplicities, and the underlying principles that govern their existence. In this comprehensive investigation, we will analyze the nature of cubic functions, the fundamental theorem of algebra, complex conjugates, and the specific conditions under which such roots may or may not coexist.

Understanding Polynomial Roots and Coefficients

A polynomial function of degree three, or a cubic function, can generally be expressed as:

$$ f(x) = ax^3 + bx^2 + cx + d $$

where $a, b, c, d \in \mathbb{R}$, and $a \neq 0$.

The roots (or zeros) of $f(x)$ are the solutions to $f(x) = 0$. These roots can be real or complex numbers. The fundamental theorem of algebra guarantees that every polynomial of degree n has exactly n roots in the complex plane, counting multiplicities.

Key points:

- The roots of a polynomial with real coefficients are either real numbers or complex conjugate pairs.
- The complex conjugate root theorem states that if a polynomial has real coefficients and a non-real root $z = p + qi$, then its conjugate $\bar{z} = p - qi$ is also a root.

Roots of Cubic Functions with Real Coefficients

Given a cubic polynomial with real coefficients, the possible configurations of roots are:

1. Three real roots (distinct or with multiplicities).
2. One real root and a pair of complex conjugate roots.

This is a direct consequence of the complex conjugate root theorem and the polynomial's degree.

Scenario 1: Three real roots

- The cubic can factor into three linear factors with real roots.
- The discriminant Δ determines the nature of roots:
- $\Delta > 0$: three distinct real roots.
- $\Delta = 0$: multiple roots, at least two roots coincide.
- $\Delta < 0$: one real root and two complex conjugate roots.

Scenario 2: One real root and one complex conjugate pair

- When $\Delta < 0$, the polynomial has exactly one real root and a pair of complex conjugates.

The Core Question: Can a Cubic Have Exactly Two Real Zeros and One Complex Zero?

The crux of the inquiry is whether a cubic polynomial with real coefficients can have exactly two real zeros and one complex zero.

Clarification:

- Roots are counted with multiplicity.
- The total number of roots, including complex roots, is three.
- The roots, when complex, appear in conjugate pairs.

Given these facts, the question reduces to: Is it possible for a cubic polynomial with real coefficients to have two roots that are real, and one root that is complex but not part of a conjugate pair?

Analysis: The Roots and Their Conjugates

Since the polynomial coefficients are real, the complex roots must come in conjugate pairs. Therefore:

- If $f(x)$ has a complex root $z = p + qi$ where $q \neq 0$, then its conjugate $\bar{z} = p - qi$ must also be a root.
- Roots with multiplicity can coincide; for example, a double real root at $x = r$, or a triple root.

Implication:

- The roots of a cubic polynomial with real coefficients cannot include a single complex root without its conjugate.
- Roots must either be:
- All three real roots, or
- One real root and a pair of complex conjugate roots.

Therefore:

> A cubic polynomial with real coefficients cannot have exactly two real zeros and a single complex zero. The complex zero must always be accompanied by its conjugate, making the total number of roots either:

- Three real roots (all real zeros),
- One real root plus a pair of complex conjugate roots.

Mathematical Justification: The Fundamental Theorem of Algebra and Conjugate Roots

The formal justification stems from the complex conjugate root theorem and the structure of polynomials with real coefficients.

Step-by-step reasoning:

1. Fundamental Theorem of Algebra:
Every polynomial of degree n has exactly n roots in the complex plane, counting multiplicities.

2. Complex conjugate root theorem:
If the coefficients are real, then non-real roots come in conjugate pairs.

3. Application to cubic functions:
Since the degree is 3:

- If there is a complex root $p + qi$ (with $q \neq 0$), then $p - qi$ must also be a root.
- The remaining root must be real (since total roots sum to 3).

4. Conclusion:

- The roots are either:
- All real (including possibly multiple roots),
- Or one real root and a conjugate pair of complex roots.

Hence, a cubic polynomial with real coefficients cannot have only two real roots and a single, non-conjugate complex root.

Examples and Non-Examples

To reinforce the theoretical findings, consider concrete examples:

Example 1: All real roots

$$ f(x) = (x - 1)(x - 2)(x - 3) $$

Roots: $1, 2, 3$ (all real).

Example 2: One real root and a conjugate pair

$$ f(x) = (x - 1)\left(x^2 + 4\right) $$

Roots: $x = 1$ (real), and $x = \pm 2i$ (complex conjugates).

Non-example:

Suppose someone claims the polynomial:

$$ g(x) = (x - 1)(x - 2) + 3i $$

which is not a polynomial with real coefficients, so it is invalid for this discussion.

Implications for Polynomial Factorization and Roots

The analysis confirms that for cubic functions with real coefficients:

- The roots' structure is constrained by conjugate symmetry.
- The presence of exactly two real zeros and one complex zero (not forming a conjugate pair) is impossible.

This has direct implications in algebra, calculus, and numerical methods, especially in root-finding algorithms and polynomial factorization.

Summary and Final Verdict

In conclusion:

- A cubic function with real coefficients cannot have exactly two real zeros and a single complex zero.
- The complex zeros must always occur in conjugate pairs; thus, the roots are either:
- Three real roots, or
- One real root and a pair of complex conjugate roots.

This is a fundamental property rooted in the complex conjugate root theorem and the nature of polynomials with real coefficients.

Therefore, the original question is answered with a definitive no:

> No, a cubic function with real coefficients cannot have two real zeros and one complex zero that is not part of a conjugate pair.

Closing Remarks

Understanding the root structures of polynomials with real coefficients is essential for many branches of mathematics and applied sciences. Recognizing that complex roots must appear in conjugate pairs simplifies the analysis and assists in graphical interpretations, solution strategies, and algebraic factorizations. This investigation underscores the importance of foundational algebraic principles and their implications in higher mathematics.

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Author's Note: This investigation not only clarifies a common misconception but also emphasizes the elegance and consistency of algebraic structures, illustrating how fundamental theorems shape our understanding of polynomial roots.

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