Exercises 2.17 Interpolate A Cubic Spline Between The Threepoints (0, 1), (2, 2) And (4, 0).

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Introduction to Cubic Spline Interpolation

Cubic spline interpolation is a powerful numerical method used to construct a smooth curve passing through a given set of points. Unlike simple polynomial interpolation, which can lead to oscillations between points (Runge's phenomenon), cubic splines ensure a smooth and stable fit by piecing together cubic polynomials with continuous first and second derivatives. This technique is particularly useful in data fitting, computer graphics, and numerical analysis, where smoothness and accuracy are essential.

In this article, we delve into the process of interpolating a cubic spline through three specific points: (0, 1), (2, 2), and (4, 0). We explore the mathematical formulation, derive the spline coefficients, and discuss the properties of the resulting spline.

Understanding the Data Points

Given Points

The data points provided for interpolation are:

1. (0, 1)
2. (2, 2)
3. (4, 0)

These points define the key locations through which the interpolating spline must pass. The goal is to find a piecewise cubic function $S(x)$ such that:

- $S(x)$ passes through all three points.
- $S(x)$ is twice continuously differentiable over the interval $[0, 4]$.
- The spline consists of two cubic polynomials: one from $x=0$ to $x=2$, and another from $x=2$ to $x=4$.

Formulating the Cubic Spline

Partitioning the Interval

Since we have three points, the interval $[0, 4]$ is divided into two subintervals:

- $[x_0, x_1] = [0, 2]$
- $[x_1, x_2] = [2, 4]$

Corresponding to these, we define two cubic polynomials:

$$
S_1(x) = a_1 + b_1(x - x_0) + c_1(x - x_0)^2 + d_1(x - x_0)^3
$$

$$
S_2(x) = a_2 + b_2(x - x_1) + c_2(x - x_1)^2 + d_2(x - x_1)^3
$$

where:

- $S_1(x)$ is valid for $x \in [0, 2]$,
- $S_2(x)$ is valid for $x \in [2, 4]$.

Conditions for the Spline

To ensure a smooth cubic spline, the following conditions must be met:

1. Interpolation Conditions:

- $S_1(0) = 1$ (passes through (0, 1))
- $S_1(2) = 2$ (passes through (2, 2))
- $S_2(2) = 2$ (continuity at $x=2$)
- $S_2(4) = 0$ (passes through (4, 0))

2. Smoothness Conditions:

- Continuity of first derivatives at $x=2$:

$$
S_1'(2) = S_2'(2)
$$

- Continuity of second derivatives at $x=2$:

$$
S_1''(2) = S_2''(2)
$$

3. Boundary Conditions:

- Since no specific boundary conditions are given, a common choice is the "natural" spline, where second derivatives at endpoints are zero:

$$
S_1''(0) = 0, \quad S_2''(4) = 0
$$

Alternatively, for simplicity, in this case, we can proceed with these natural boundary conditions.

Deriving the Coefficients

Step 1: Expressing the Cubic Polynomials

Let’s set:

- $x_0 = 0$, $x_1 = 2$, $x_2 = 4$.

The cubic polynomials:

$$
S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3
$$

$$
S_2(x) = a_2 + b_2 (x - 2) + c_2 (x - 2)^2 + d_2 (x - 2)^3
$$

Note: Using shifted variables for S_2 simplifies the calculations.

Step 2: Applying the Interpolation Conditions

From $S_1(0) = 1$:

$$
a_1 + b_1 \cdot 0 + c_1 \cdot 0 + d_1 \cdot 0 = 1 \Rightarrow a_1 = 1
$$

From $S_1(2) = 2$:

$$
a_1 + 2b_1 + 4c_1 + 8d_1 = 2
$$

But since $a_1 = 1$:

$$
1 + 2b_1 + 4c_1 + 8d_1 = 2 \Rightarrow 2b_1 + 4c_1 + 8d_1 = 1
$$

From $S_2(2) = 2$:

At $x=2$, $x - 2=0$:

$$
a_2 + b_2 \cdot 0 + c_2 \cdot 0 + d_2 \cdot 0 = 2 \Rightarrow a_2=2
$$

\]

From \(S_2(4) = 0 \):

At \(x=4 \), \(x - 2=2 \):

\[
a_2 + b_2 \cdot 2 + c_2 \cdot 4 + d_2 \cdot 8 = 0
\]

Using \(a_2=2 \):

\[
2 + 2b_2 + 4c_2 + 8d_2= 0
\]

Step 3: Derivative Conditions for Smoothness

First derivatives:

\[
S_1'(x) = b_1 + 2c_1 x + 3d_1 x^2
\]

\[
S_2'(x) = b_2 + 2c_2 (x - 2) + 3d_2 (x - 2)^2
\]

At \(x=2 \):

\[
S_1'(2) = b_1 + 4 c_1 + 12 d_1
\]

\[
S_2'(2) = b_2 + 0 + 0 = b_2
\]

Set equal for smoothness:

\[
b_1 + 4 c_1 + 12 d_1 = b_2
\]

Second derivatives:

\[
S_1''(x) = 2 c_1 + 6 d_1 x
\]

\[
S_2''(x) = 2 c_2 + 6 d_2 (x - 2)
\]

At \(x=2 \):

\[

$$
S_1''(2) = 2 c_1 + 12 d_1
$$

$$
S_2''(2) = 2 c_2 + 0 = 2 c_2
$$

Set equal:

$$
2 c_1 + 12 d_1 = 2 c_2
$$

Step 4: Boundary Conditions for Natural Spline

At $x=0$:

$$
S_1''(0) = 2 c_1 + 0 = 0 \Rightarrow c_1=0
$$

At $x=4$:

$$
S_2''(4) = 2 c_2 + 6 d_2 \cdot 2 = 2 c_2 + 12 d_2=0
$$

Solving the System for Coefficients

Now, compile all the equations:

1. $a_1=1$

2. $2b_1 + 4 c_1 + 8 d_1=1$

3. $a_2=2$

4. $2 + 2b_2 + 4 c_2 + 8 d_2=0$

5. $b_2 = b_1 + 4 c_1 + 12 d_1$

6. $2 c_1 + 12 d_1= 2 c_2$

7. \(

Frequently Asked Questions

What is the primary goal when interpolating a cubic spline through the points (0, 1), (2, 2), and (4, 0)?

The primary goal is to construct a smooth, continuous function composed of cubic polynomial segments that exactly pass through the given points, ensuring a smooth transition between them without oscillations.

How many cubic polynomial segments are needed to interpolate the points (0, 1), (2, 2), and (4, 0)?

Two cubic polynomial segments are needed: one between (0, 1) and (2, 2), and another between (2, 2) and (4, 0).

What boundary conditions are typically used when constructing a natural cubic spline for these points?

Natural cubic splines usually impose zero second derivatives at the endpoints, meaning the second derivative at (0, 1) and (4, 0) are set to zero to ensure a smooth, 'free' boundary.

Can you briefly outline the steps to interpolate these points using a cubic spline?

First, set up the system of equations enforcing the spline passes through the points and has continuous first and second derivatives at the internal point (2, 2). Then, apply boundary conditions (e.g., natural spline), solve for the spline coefficients, and finally, define the cubic polynomials for each segment.

What are the advantages of using cubic splines over simpler interpolation methods like linear interpolation for these points?

Cubic splines provide a smooth and visually appealing curve with continuous first and second derivatives, avoiding the sharp corners or oscillations common in linear interpolation, leading to more realistic and natural-looking interpolations.

Additional Resources

Exercises 2.17 Interpolate A Cubic Spline Between The Three Points (0, 1), (2, 2), and (4, 0)

Cubic spline interpolation is a fundamental technique in numerical analysis and computer graphics, used to generate smooth curves passing through a given set of data points. Among the various interpolation methods, cubic splines are particularly favored because they provide a smooth, continuous, and differentiable curve that closely follows the data while avoiding the oscillations common with high-degree polynomials. Exercise 2.17 focuses on constructing a cubic spline passing through three specific points: (0, 1), (2, 2), and (4, 0). This task offers a practical application of spline theory, emphasizing how to determine the spline's coefficients, enforce smoothness conditions, and interpret the resulting curve.

Understanding the Fundamentals of Cubic Spline Interpolation

Before delving into the specifics of the exercise, it is essential to grasp the core concepts behind cubic spline interpolation.

What Is a Cubic Spline?

A cubic spline is a piecewise-defined function composed of cubic polynomials, one for each interval between the data points, that together form a smooth curve. These polynomials are constructed to satisfy the following conditions:

- Interpolation Condition: The spline passes through all given data points.
- Continuity Conditions: The first and second derivatives of adjacent polynomial segments are equal at the interior points, ensuring smoothness.
- Boundary Conditions: Additional conditions at the endpoints (such as natural, clamped, or not-a-knot) specify the behavior at the boundaries.

Why Use Cubic Splines?

Compared to simple polynomial interpolation, cubic splines have several advantages:

- Smoothness: They guarantee continuous first and second derivatives.
- Local Control: Changes in one segment do not drastically affect others.
- Reduced Oscillations: Unlike high-degree polynomials, splines avoid Runge’s phenomenon, which causes oscillations at the edges.

Step-by-Step Guide to Interpolating a Cubic Spline for the Given Points

In this exercise, the goal is to construct a cubic spline that passes through the three points: (0, 1), (2, 2), and (4, 0). The process involves several stages, including setting up equations for the spline segments and solving for their coefficients.

1. Define the Spline Segments

Since there are three points, the spline consists of two cubic polynomial segments:

- $S_1(x)$ for $x \in [0, 2]$
- $S_2(x)$ for $x \in [2, 4]$

Each segment can be written as:

$$
S_i(x) = a_i + b_i (x - x_i) + c_i (x - x_i)^2 + d_i (x - x_i)^3
$$

where (x_i, y_i) are the data points.

For simplicity, define:

$$
S_1(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3, \quad x \in [0, 2]
$$
$$
S_2(x) = a_1 + b_1 (x - 2) + c_1 (x - 2)^2 + d_1 (x - 2)^3, \quad x \in [2, 4]
$$

Establishing Conditions for the Spline

The next step is to set up equations based on the interpolation and smoothness conditions.

2. Interpolation Conditions

The spline must pass through the data points:

$$
S_1(0) = 1 \Rightarrow a_0 = 1
$$
$$
S_1(2) = 2 \Rightarrow a_0 + 2b_0 + 4 c_0 + 8 d_0 = 2
$$
$$
S_2(2) = 2 \Rightarrow a_1 = 2
$$
$$
S_2(4) = 0 \Rightarrow a_1 + 2b_1 + 4 c_1 + 8 d_1 = 0
$$

Note that since $S_1(2)$ and $S_2(2)$ must be equal at the junction point (2,2), this provides a continuity condition.

3. Smoothness Conditions (Continuity of Derivatives)

The first and second derivatives must be continuous at $x=2$:

$$
S_1'(2) = S_2'(2)
$$
$$
S_1''(2) = S_2''(2)
$$

Expressed in terms of coefficients:

$$
b_0 + 2 c_0 (2) + 3 d_0 (2)^2 = b_1
$$

$$
2 c_0 + 6 d_0 (2) = 2 c_1
$$

Solving for the Coefficients

With the established equations, the next step involves solving for the unknown coefficients $b_0, c_0, d_0, b_1, c_1, d_1$.

4. Boundary Conditions

Since the problem does not specify boundary conditions at the endpoints, the natural spline boundary condition (second derivatives at endpoints are zero) is a common choice:

$$
S_1''(0) = 2 c_0 = 0 \Rightarrow c_0 = 0
$$
$$
S_2''(4) = 2 c_1 + 6 d_1 (4 - 2) = 0
$$

This introduces additional equations, which, combined with previous conditions, allow us to solve for all coefficients systematically.

5. Final Calculations

By substituting known values and solving the resulting linear system, you obtain explicit expressions for each coefficient. The process can be performed by hand for such a small system or using computational tools like MATLAB, NumPy, or other linear algebra software.

Features, Pros, and Cons of This Approach

Features:

- Constructing a cubic spline involves solving a small linear system, making it computationally efficient.
- The method guarantees a smooth curve passing through all points.
- Boundary conditions can be adapted (natural, clamped, or not-a-knot) depending on the application.

Pros:

- Produces visually appealing, smooth interpolations suitable for data visualization.
- Local control: changes in data affect only their neighboring segments.
- Well-suited for data with no oscillations or sharp discontinuities.

Cons:

- Requires solving linear systems, which can become computationally intensive for large datasets.
- Boundary conditions influence the shape of the spline at the edges; choosing inappropriate conditions can distort the interpolation.
- Not ideal for data with noise unless modifications are made (e.g., smoothing splines).

Interpreting the Resulting Spline

Once the coefficients are determined, the spline functions $S_1(x)$ and $S_2(x)$ fully describe the interpolated curve. Plotting these functions reveals a smooth trajectory passing through the three points, with continuous first and second derivatives. This can be used for:

- Data visualization to understand underlying trends.
- Generating intermediate points for animations or simulations.
- Serving as a basis for further analysis, such as derivative estimation or optimization.

Conclusion and Practical Insights

Interpolating a cubic spline between three points demonstrates the elegance and utility of spline theory in numerical analysis. The process underscores the importance of setting up the correct system of equations based on interpolation, smoothness, and boundary conditions, then solving for the polynomial coefficients. The resulting spline is a powerful tool for creating smooth, visually appealing curves that honor the original data points while maintaining mathematical smoothness.

For practitioners, mastering this exercise enhances understanding of spline mechanics, prepares them for more complex interpolation tasks involving larger datasets, and highlights the importance of boundary conditions in shaping the interpolated curve. While straightforward for small datasets, the principles extend seamlessly to larger problems, reinforcing the significance of cubic splines in computational mathematics, engineering, and computer graphics.

In summary:

- Cubic splines provide smooth and flexible interpolation.
- The process involves defining piecewise polynomial segments and enforcing constraints.
- Boundary conditions are crucial in shaping the final curve.
- The method balances computational efficiency with high-quality curve modeling.
- Practical applications span data fitting, animation, and scientific computing.

By mastering Exercise 2.17, students and practitioners deepen their understanding of spline interpolation, equipping them with a versatile toolset for a broad array of mathematical and engineering challenges.

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