Find A Polar Equation For The Curve Represented By The Given Cartesian Equation. Xy = 1

Find A Polar Equation For The Curve Represented By The Given Cartesian Equation. Xy = 1

Understanding how to convert equations from Cartesian to polar coordinates is a fundamental skill in analytical geometry. The Cartesian equation xy = 1 describes a hyperbola, and expressing this curve in polar form provides valuable insights into its geometry, symmetry, and properties. This article offers a comprehensive guide to deriving the polar equation from the Cartesian form xy = 1, including step-by-step procedures, explanations of key concepts, and practical applications. Whether you're a student studying conic sections or a mathematician working on advanced curve analysis, mastering this conversion enhances your mathematical toolkit.

Understanding Cartesian and Polar Coordinates

Before diving into the conversion process, it's essential to understand the basics of Cartesian and polar coordinate systems.

Cartesian Coordinates

- Defined by an ordered pair (x, y).
- The x-coordinate indicates horizontal position.
- The y-coordinate indicates vertical position.
- Used for plotting points in a rectangular grid.

Polar Coordinates

- Defined by a radius r and an angle θ.
- The point P is represented as (r, θ).
- r is the distance from the origin to the point.
- θ is the angle measured counterclockwise from the positive x-axis.
- Conversion formulas:
- $x = r \cos \theta$
- $y = r \sin \theta$

Understanding these relationships is crucial for transforming equations between the two systems.

Given Cartesian Equation: $xy = 1$

The Cartesian equation $xy = 1$ describes a rectangular hyperbola, which is symmetric with respect to the lines $y = x$ and $y = -x$. This hyperbola is located in the first and third quadrants, where both x and y are positive or both are negative.

Key features of the hyperbola $xy = 1$:
- As x approaches 0, y approaches infinity (or negative infinity).
- As x increases, y decreases accordingly.
- The hyperbola has two branches, one in quadrants I and III.

The goal is to find an equivalent expression of this curve in polar coordinates.

Converting Cartesian Equation $xy = 1$ to Polar Coordinates

The process involves replacing x and y with their polar equivalents and simplifying the resulting expression.

Step 1: Recall the Conversion Formulas

- $x = r \cos \theta$
- $y = r \sin \theta$

Step 2: Substitute x and y into the Cartesian Equation

Given $xy = 1$:
- $(r \cos \theta)(r \sin \theta) = 1$

Step 3: Simplify the Expression

- $r^2 \cos \theta \sin \theta = 1$

This is the core polar form of the hyperbola $xy = 1$.

Step 4: Express r^2 in terms of θ

- $r^2 = 1 / (\cos \theta \sin \theta)$

This formula represents the hyperbola in polar coordinates, with the domain of θ restricted to values where the denominator is non-zero.

Analyzing the Polar Equation $r^2 = 1 / (\cos \theta \sin \theta)$

The derived equation:
- $r^2 = 1 / (\cos \theta \sin \theta)$

has important implications and properties:

Domain considerations:
- The equation is valid where $\cos \theta \sin \theta \neq 0$.
- $\cos \theta = 0$ at $\theta = \pi/2, 3\pi/2$ (vertical lines).
- $\sin \theta = 0$ at $\theta = 0, \pi$ (horizontal lines).

Therefore, the hyperbola has asymptotes along these lines, and the polar equation is undefined at these angles, indicating the presence of asymptotes.

Interpreting the Polar Equation

Understanding the behavior of r as a function of θ is essential:

- When θ approaches 0 or π, r approaches infinity, indicating the hyperbola's branches extend toward these angles.
- When θ approaches π/2 or 3π/2, r approaches zero or infinity, reflecting the hyperbola's asymptotic behavior.

The equation demonstrates the hyperbola's symmetry about the lines $y = x$ and $y = -x$, consistent with its Cartesian form.

Graphing the Hyperbola in Polar Coordinates

To visualize the curve:

1. Select values of θ within the domain where the equation is defined.
2. Calculate r using $r^2 = 1 / (\cos \theta \sin \theta)$.
3. Plot the points (r, θ) in polar coordinates.
4. Recognize the asymptotes along $\theta = 0, \pi/2, \pi$, and $3\pi/2$.

Practical tips:
- Use a graphing calculator or software with polar plotting capabilities.

- Pay attention to the asymptotes and the behavior of r near the singularities.

Alternative Forms and Simplifications

The derived equation can be manipulated further:

Expressing r in terms of θ:
- $r = \pm 1 / \sqrt{(\cos \theta \sin \theta)}$

Using double angle identities:
- Recall that $\sin 2\theta = 2 \sin \theta \cos \theta$
- Therefore, $\cos \theta \sin \theta = (1/2) \sin 2\theta$

Substitute into the equation:

$r^2 = 1 / [(1/2) \sin 2\theta] = 2 / \sin 2\theta$

Thus, the polar equation can also be written as:

$r^2 = 2 / \sin 2\theta$

This form is often more convenient for analysis and graphing.

Key Properties of the Polar Hyperbola

Understanding the properties of the curve in polar form enhances comprehension:

- Asymptotes: Along $\theta = 0, \pi/2, \pi, 3\pi/2$ where r tends to infinity.
- Symmetry: The curve exhibits symmetry about lines $y = x$ and $y = -x$.
- Branches: The hyperbola has two branches, corresponding to positive and negative r values.
- Behavior near asymptotes: r becomes very large or very small, indicating the curve approaches the asymptotes.

Applications of Converting Cartesian Equations to Polar Form

Converting equations like $xy = 1$ into polar form is useful across various fields:

- Physics: Analyzing electromagnetic fields or wave patterns with symmetrical properties.
- Engineering: Designing components with hyperbolic geometries.
- Mathematics: Studying conic sections, their properties, and their transformations.
- Computer Graphics: Rendering curves and surfaces in polar coordinates.

Practice Problems and Examples

To solidify understanding, consider practicing with these problems:

1. Find the polar equation of the circle $x^2 + y^2 = 4$.
2. Convert the Cartesian equation $y = x^2$ into polar form.
3. Sketch the curve $xy = 1$ in polar coordinates and analyze its asymptotes.

Summary and Conclusion

Transforming the Cartesian equation $xy = 1$ into its polar form involves straightforward substitutions and algebraic manipulations. The key result:

$r^2 = 1 / (\cos\theta \sin\theta)$

or equivalently,

$r^2 = 2 / \sin 2\theta$

provides a powerful representation of the hyperbola in the polar coordinate system. This form reveals the curve's symmetry, asymptotes, and behavior near singularities. Understanding this conversion not only deepens geometric intuition but also enhances analytical skills in advanced mathematics. Whether for academic purposes, research, or practical applications, mastering the process of converting Cartesian equations into polar form is an invaluable mathematical tool.

Keywords: polar equation, Cartesian to polar conversion, hyperbola xy=1, conic sections, polar coordinates, hyperbola properties, asymptotes, mathematical transformation, curve analysis, analytical geometry

Frequently Asked Questions

What is the polar form of the Cartesian equation xy = 1?

The polar form of xy = 1 is $r^2 \sin(\theta) \cos(\theta) = 1$, which can also be written as $r^2 (1/2) \sin(2\theta) = 1$, or $r^2 = 2 / \sin(2\theta)$.

How do you convert the Cartesian equation xy = 1 into a polar equation?

Replace x with $r \cos(\theta)$ and y with $r \sin(\theta)$, then multiply: $(r \cos(\theta))(r \sin(\theta)) = 1$, resulting in $r^2 \cos(\theta) \sin(\theta) = 1$. Using the identity $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$, the equation becomes $r^2 (1/2) \sin(2\theta) = 1$, or $r^2 = 2 / \sin(2\theta)$.

What type of curve is represented by the polar equation $r^2 = 2 / \sin(2\theta)$?

This polar equation represents a lemniscate-like curve, specifically a rectangular hyperbola in Cartesian coordinates, with symmetry about the axes.

Are there any special points or asymptotes associated with the polar curve $r^2 = 2 / \sin(2\theta)$?

Yes, the curve has asymptotes where $\sin(2\theta)$ approaches zero, i.e., at $\theta = 0, \pi/2, \pi$, etc., corresponding to the lines where the denominator approaches zero, leading to the curve extending towards infinity.

Can the equation xy = 1 be written in a form to easily plot in polar coordinates?

Yes, by expressing it as $r^2 = 2 / \sin(2\theta)$, you can plot the curve in polar coordinates by calculating r for various θ values, noting the asymptotes where $\sin(2\theta) = 0$.

Additional Resources

Polar Equation Derivation for the Cartesian Curve xy = 1: A Comprehensive Exploration

---

Introduction

When delving into the fascinating world of coordinate systems, the interplay between Cartesian and polar equations often reveals deeper insights into the geometry of curves. The Cartesian equation xy = 1 offers a compelling case study: a hyperbola with specific asymptotic behaviors and symmetry properties. Converting this curve into its polar form not only enhances our understanding of its geometric structure but also broadens our toolkit for analyzing complex curves in different coordinate systems. In this article, we undertake an in-depth exploration of deriving the polar equation corresponding to the Cartesian curve xy = 1, presenting a step-by-step methodology, key insights, and implications that serve as a detailed guide for students, educators, and enthusiasts alike.

Understanding the Cartesian Equation: xy = 1

The Nature of the Curve

The Cartesian equation:

$$
xy = 1
$$

represents a classic hyperbola centered at the origin. Unlike the standard hyperbolas oriented along the x- or y-axis, this particular form displays symmetry with respect to the origin, with its branches located in the quadrants where x and y have opposite signs (Quadrants II and IV).

Key features of the curve:

- Asymptotes: The lines $x=0$ and $y=0$ serve as asymptotes because as $x \to 0$, $y \to \infty$, and vice versa.
- Symmetry: The curve is symmetric about the origin, which reflects the fact that replacing (x,y) with $(-x,-y)$ leaves the equation unchanged.
- Behavior at infinity: As $|x| \to \infty$, $|y| \to 0$, and vice versa.
- Branches: The hyperbola has two branches in the second and fourth quadrants.

Understanding these features prepares us for an effective transition into polar coordinates, where the behavior of the curve can be visualized through the relation between the radius r and the angle θ.

Transitioning from Cartesian to Polar Coordinates

The Foundations of Polar Coordinates

Recall that in polar coordinates:

$$
x = r \cos \theta
$$
$$
y = r \sin \theta
$$

where:

- $r \geq 0$ is the distance from the origin to a point (x,y),
- θ is the angle measured from the positive x-axis to the line segment connecting the origin to the point.

The goal is to express the Cartesian equation $xy = 1$ in terms of r and θ, yielding the polar form $r = f(\theta)$.

Deriving the Polar Equation: Step-by-Step

Step 1: Substituting Cartesian to Polar Expressions

Starting with the Cartesian equation:

$$
xy = 1
$$

Substitute $x = r \cos \theta$ and $y = r \sin \theta$:

$$
(r \cos \theta)(r \sin \theta) = 1
$$
$$
r^2 \cos \theta \sin \theta = 1
$$

This simplifies to:

$$
r^2 (\sin \theta \cos \theta) = 1
$$

Step 2: Isolating r^2

Rearranged, the equation becomes:

$$
r^2 = \frac{1}{\sin \theta \cos \theta}
$$

Provided that $\sin \theta \cos \theta \neq 0$, the relation holds.

Recognizing that $\sin \theta \cos \theta$ can be expressed using a double-angle identity, we proceed to simplify further.

Step 3: Applying Trigonometric Identities

The double-angle identity:

$$
\sin 2\theta = 2 \sin \theta \cos \theta
$$

enables us to write:

$$
\sin \theta \cos \theta = \frac{\sin 2\theta}{2}
$$

Thus, the expression for r^2 becomes:

$$
r^2 = \frac{1}{\frac{\sin 2\theta}{2}} = \frac{2}{\sin 2\theta}
$$

To find r:

$$
r = \pm \sqrt{\frac{2}{\sin 2\theta}}
$$

This is the polar equation for the curve $xy=1$.

Final Polar Equation and Its Interpretation

$$
\boxed{
r(\theta) = \pm \sqrt{\frac{2}{\sin 2\theta}}
}
$$

This equation describes the radius r as a function of the angle θ, with the understanding that:

- The $\pm$ accounts for the two branches of the hyperbola.
- The domain is constrained by the condition $\sin 2\theta \neq 0$, i.e., $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, etc., where the denominator becomes zero, indicating asymptotes or undefined points.

Geometric and Analytical Insights

Behavior of the Polar Equation

- Asymptotes: When $\sin 2\theta \to 0$, r tends to infinity, corresponding to the asymptotic lines of the hyperbola. Specifically, at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, the function approaches infinity or is undefined, mirroring the Cartesian asymptotes at $x=0$ and $y=0$.

- Symmetry: The curve exhibits symmetry with respect to the origin, consistent with the $\pm$ in the expression.
- Quadrant behavior: The sign of $\sin 2\theta$ determines which branch of the hyperbola is traced for a given θ.

Plotting Considerations

Plotting this polar equation involves selecting θ values in intervals that avoid the asymptotes, calculating corresponding r, and then plotting the points (r, θ). The hyperbola's branches appear in quadrants II and IV, matching the Cartesian description.

Broader Implications and Applications

Why Convert to Polar?

Converting the hyperbola $xy=1$ into polar form is more than a mere algebraic exercise; it unlocks various applications:

- Enhanced visualization: Polar plots can more intuitively display the asymptotic behavior and symmetry.
- Integration and calculus: Certain integrals involving the curve are more straightforward in polar coordinates.
- Physics and engineering: Many problems involving radial symmetry or angular dependencies benefit from polar representations.

Limitations and Domain Restrictions

While the derived polar equation provides a powerful tool, practitioners must be cautious:

- Undefined points: At angles where $\sin 2\theta = 0$, the equation is undefined, corresponding to asymptotes.
- Sign considerations: The $\pm$ indicates two branches; choosing the sign depends on the quadrant and the specific branch being analyzed.

Summary: Key Takeaways

- The Cartesian hyperbola $xy=1$ can be converted into polar coordinates, resulting in:

$$
r(\theta) = \pm \sqrt{\frac{2}{\sin 2\theta}}
$$

- This form elegantly encodes the curve's asymptotic behavior and symmetry.
- The conversion leverages fundamental trigonometric identities, showcasing the interconnectedness of algebra and geometry.

Conclusion

Transforming the Cartesian equation $xy=1$ into its polar form exemplifies the power of coordinate system conversions in advanced mathematics. This derivation not only deepens conceptual understanding but also provides practical tools for plotting, analysis, and problem-solving involving hyperbolic curves. Whether for academic pursuits or applied physics, mastering these conversions enriches one's geometric intuition and analytical versatility.

In essence, the polar equation $r = \pm \sqrt{\frac{2}{\sin 2\theta}}$ stands as a testament to the elegance and utility of coordinate transformations, bridging the gap between algebraic form and geometric intuition.

Find A Polar Equation For The Curve Represented By The Given Cartesian Equation Xy 1

Find A Polar Equation For The Curve Represented By The Given Cartesian Equation Xy 1

Related Articles

- The Function C(x) = 18x -10 Represents The Cost In Dollars. How Many Tickets Can You Buy?
- Which sentences best illustrate the narrator’s conflict? in blood of moles
- Explain Why A Sterospecific Active Site Can Only Bond To One Enatomeric Form A Substrate

Back to Home