Find All Complex Cube Roots Of 4+i. Give Your Answers In A+bi Form, Separated By Commas.

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Introduction to Complex Cube Roots

In the fascinating realm of complex numbers, finding roots of complex expressions is a fundamental task that combines algebra, geometry, and trigonometry. Among these, computing cube roots of complex numbers is particularly interesting because each complex number, except zero, has exactly three cube roots in the complex plane. These roots are evenly spaced in argument (angle) and can be expressed in the form a + bi, where a and b are real numbers.

Today, we focus on finding all the cube roots of a specific complex number: 4 + i. The goal is to determine all solutions to the equation:

$$ z^3 = 4 + i $$

and express each root in the standard a + bi form.

Understanding the Problem

Before diving into calculations, it's important to understand the key concepts involved:

- Complex numbers are expressed as a + bi, where a is the real part, and b is the imaginary part.
- Modulus (r) of a complex number $z = a + bi$ is given by:

$$ r = |z| = \sqrt{a^2 + b^2} $$

- Argument (θ) is the angle between the positive real axis and the line segment from the origin to the point (a, b):

$$ \theta = \arg(z) = \arctan\left(\frac{b}{a}\right) $$

- De Moivre's Theorem states that for a complex number in polar form $z = r(\cos \theta + i \sin \theta)$, its n-th roots are given by:

$$z_k = r^{1/n} \left(\cos \left(\frac{\theta + 2k\pi}{n}\right) + i \sin \left(\frac{\theta + 2k\pi}{n}\right) \right)$$

for $k = 0, 1, 2, ..., n-1$.

Applying these principles, our task involves converting 4 + i into polar form, determining its modulus and argument, and then applying De Moivre's theorem to find all cube roots.

Step 1: Convert 4 + i to Polar Form

The first step is to compute the modulus r and argument θ.

Calculating the Modulus r

$$r = \sqrt{(4)^2 + (1)^2} = \sqrt{16 + 1} = \sqrt{17}$$

Calculating the Argument θ

$$\theta = \arctan \left(\frac{1}{4} \right)$$

Since $\frac{1}{4}$ is a small positive number, θ is approximately:

$$\theta \approx \arctan(0.25) \approx 0.244978 \text{ radians}$$

Expressed in degrees, this is roughly 14.04°, but for our calculations, we'll retain radians for precision.

Polar form of 4 + i

$$4 + i = r (\cos \theta + i \sin \theta) = \sqrt{17} \left(\cos 0.244978 + i \sin 0.244978 \right)$$

Step 2: Apply De Moivre's Theorem to Find Cube Roots

The general formula for the cube roots of a complex number $z = r (\cos \theta + i \sin \theta)$ is:

$$ z_k = r^{1/3} \left(\cos \left(\frac{\theta + 2k\pi}{3} \right) + i \sin \left(\frac{\theta + 2k\pi}{3} \right) \right) $$

for $k = 0, 1, 2$.

Compute $r^{1/3}$

$$ r^{1/3} = (\sqrt{17})^{1/3} = 17^{1/6} $$

which is approximately:

$$ 17^{1/6} \approx e^{\frac{1}{6} \ln 17} $$

Calculating:

$$ \ln 17 \approx 2.8332 $$

$$ \frac{1}{6} \times 2.8332 \approx 0.4722 $$

$$ e^{0.4722} \approx 1.603 $$

Thus:

$$ r^{1/3} \approx 1.603 $$

Calculate the three arguments

For each root z_k, the argument is:

$$ \theta_k = \frac{\theta + 2k\pi}{3} $$

where $k = 0, 1, 2$.

- For $k = 0$:

$$ \theta_0 = \frac{0.244978 + 0}{3} = 0.08166 \text{ radians} $$

- For $k = 1$:

$$ \theta_1 = \frac{0.244978 + 2\pi}{3} = \frac{0.244978 + 6.283185}{3} \approx \frac{6.52816}{3} \approx 2.17605 \text{ radians} $$

- For $k = 2$:

$$ \theta_2 = \frac{0.244978 + 4\pi}{3} = \frac{0.244978 + 12.56637}{3} \approx \frac{12.81135}{3} \approx 4.27045 \text{ radians} $$

Step 3: Write the Roots in a + bi Form

Now, for each k, compute:

$$z_k = r^{1/3} (\cos \theta_k + i \sin \theta_k)$$

with $r^{1/3} \approx 1.603$.

Root z_0 (when $k=0$)

$$z_0 \approx 1.603 (\cos 0.08166 + i \sin 0.08166)$$

Calculating:

$$\cos 0.08166 \approx 0.9967$$
$$\sin 0.08166 \approx 0.0816$$

Therefore:

$$z_0 \approx 1.603 \times 0.9967 + i \times 1.603 \times 0.0816$$
$$\approx 1.597 + 0.131$$

Root z_1 (when $k=1$)

$$z_1 \approx 1.603 (\cos 2.17605 + i \sin 2.17605)$$

Calculating:

$$\cos 2.17605 \approx -0.573$$
$$\sin 2.17605 \approx 0.819$$

Thus:

$$z_1 \approx 1.603 \times (-0.573) + i \times 1.603 \times 0.819$$
$$\approx -0.918 + 1.312$$

Root z_2 (when $k=2$)

$$z_2 \approx 1.603 (\cos 4.27045 + i \sin 4.27045)$$

Calculating:

$$\cos 4.27045 \approx -0.425$$
$$\sin 4.27045 \approx -0.905$$

Therefore:

$$z_2 \approx 1.603 \times (-0.425) + i \times 1.603 \times (-0.905)$$
$$\approx -0.681 - 1.45$$

Final Answers in a + bi Form

Summarizing the roots:

1. $z_0 \approx 1.597 + 0.131i$
2. $z_1 \approx -0.918 + 1.312i$
3. $z_2 \approx -0.681 - 1.45i$

Expressed precisely with rounded approximations, the three cube roots of $4 + i$ are:

Approximately:

$$1.597 + 0.131i, \, -0.918 + 1.312i, \, -0.681 - 1.45i$$

Exact forms involve radicals and trigonometric functions:

$$
z_k = 17^{1/6} \left(\cos \left(\frac{\arctan(1/4) + 2k\pi}{3} \right) + i \sin \left(\frac{\arctan(1/4) + 2k\pi}{3} \right) \right), \quad k=0,1,2
$$

Conclusion

Finding all complex cube roots of a given complex number requires converting the number into polar form, applying De Moivre's theorem, and converting back to

Frequently Asked Questions

How do I find all complex cube roots of 4 + i?

To find all cube roots of 4 + i, convert 4 + i to polar form, find its magnitude and argument, then apply De Moivre's theorem to compute the roots, resulting in three solutions in A + bi form.

What is the polar form of 4 + i used for finding

cube roots?

The polar form expresses 4 + i as r(cos θ + i sin θ), where $r = \sqrt{(4^2 + 1^2)} = \sqrt{17}$, and θ = arctangent(1/4).

How do I compute the cube roots once I have the polar form?

Apply De Moivre's theorem: each root is given by (r)^{1/3} times [cos((θ + 2πk)/3) + i sin((θ + 2πk)/3)] for k = 0, 1, 2.

What are the approximate values of the cube roots of 4 + i in A + bi form?

The approximate roots are 1.29 + 0.17i, -0.65 + 1.14i, and -0.64 - 1.31i, but exact forms involve radicals and trigonometric functions.

Why do we get three cube roots for a complex number like 4 + i?

Because complex numbers have three cube roots (including complex conjugates), corresponding to the three solutions obtained from De Moivre's theorem by adding 2πk/3 to the argument.

Can you provide the exact algebraic form of the cube roots of 4 + i?

Yes, by calculating r^{1/3} and the angles, the roots are approximately 1.29 + 0.17i, -0.65 + 1.14i, and -0.64 - 1.31i, expressed in A + bi form.

Additional Resources

Find All Complex Cube Roots of 4 + i

Understanding how to find the cube roots of a complex number is a fundamental aspect of complex analysis and algebra. This problem involves determining all complex numbers \(z \) such that \(z^3 = 4 + i \). These roots are critical in various fields, including engineering, physics, and mathematics, especially in solving polynomial equations in the complex domain, analyzing roots of unity, and understanding the geometric interpretation of complex functions.

In this comprehensive guide, we will explore the process of finding all cube roots of \(4 + i \), providing a detailed explanation of the methods involved, the mathematical principles underpinning the process, and the explicit roots expressed in \(a + bi \) form.

Understanding the Problem: Cube Roots of a Complex Number

The task is to find all complex numbers $z = a + bi$ such that:

$$
z^3 = 4 + i
$$

Here, a and b are real numbers. Since complex numbers are represented in the form $a + bi$, the goal is to determine all such pairs (a, b).

Key points:

- The equation $z^3 = 4 + i$ has exactly three solutions in the complex plane, according to the Fundamental Theorem of Algebra.
- These roots can be expressed in polar form, which simplifies computations involving powers and roots.

Step 1: Expressing $4 + i$ in Polar Form

Before attempting to find the roots, it is often easiest to convert $4 + i$ into polar (trigonometric) form:

$$
r (\cos \theta + i \sin \theta)
$$

where:

- $r = |4 + i|$ is the modulus,
- $\theta = \arg(4 + i)$ is the argument (angle).

Calculating the modulus r:

$$
r = \sqrt{(\text{Re}(4 + i))^2 + (\text{Im}(4 + i))^2} = \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}
$$

Calculating the argument θ:

\[
\theta = \arctan\left(\frac{\text{Im}}{\text{Re}}\right) =
\arctan\left(\frac{1}{4}\right)
\]

Since \(4 + i \) is in the first quadrant (positive real and imaginary parts), the principal argument is:

\[
\theta = \arctan\left(\frac{1}{4}\right)
\]

Calculating this angle:

\[
\theta \approx \arctan(0.25) \approx 0.244978 \text{ radians}
\]

Summary:

\[
4 + i = \sqrt{17} \left(\cos 0.245 + i \sin 0.245 \right)
\]

Step 2: Applying De Moivre's Theorem to Find Cube Roots

De Moivre's theorem states that for a complex number in polar form:

\[
z = r (\cos \theta + i \sin \theta)
\]

its \(n \)-th roots are given by:

\[
z_k = r^{1/n} \left(\cos \left(\frac{\theta + 2\pi k}{n} \right) + i \sin
\left(\frac{\theta + 2\pi k}{n} \right) \right), \quad k = 0, 1, ..., n-1
\]

In our case, \(n = 3 \). Therefore, the three cube roots are:

\[
z_k = \sqrt[3]{r} \left(\cos \left(\frac{\theta + 2\pi k}{3} \right) + i
\sin \left(\frac{\theta + 2\pi k}{3} \right) \right)
\]

Calculating $\sqrt[3]{r}$:

$$
\sqrt[3]{\sqrt{17}} = (\sqrt{17})^{1/3} = 17^{1/6}
$$

This is the cube root of $\sqrt{17}$, which numerically is approximately:

$$
17^{1/6} \approx e^{\frac{1}{6} \ln 17} \approx e^{\frac{1}{6} \times 2.8332} \approx e^{0.4722} \approx 1.603
$$

Step 3: Computing the Three Roots Explicitly

Now, for $k = 0, 1, 2$, compute the roots.

Root 1: $k = 0$

$$
z_0 = 1.603 \left(\cos \frac{0.245 + 0}{3} + i \sin \frac{0.245 + 0}{3} \right)
$$
$$
z_0 = 1.603 \left(\cos 0.0817 + i \sin 0.0817 \right)
$$

Calculating:

- $\cos 0.0817 \approx 0.9967$
- $\sin 0.0817 \approx 0.0816$

Thus:

$$
z_0 \approx 1.603 \times 0.9967 + i \times 1.603 \times 0.0816
$$
$$
z_0 \approx 1.597 + 0.131 i
$$

Root 2: $k = 1$

$$
z_1 = 1.603 \left(\cos \frac{0.245 + 2\pi}{3} + i \sin \frac{0.245 + 2\pi}{3} \right)
$$

$$
z_1 = 1.603 \left(\cos \frac{0.245 + 6.283}{3} \right) + i \sin \left(\frac{0.245 + 6.283}{3} \right)
$$

$$
z_1 = 1.603 \left(\cos 2.109 + i \sin 2.109 \right)
$$

Calculating:

- $\cos 2.109 \approx -0.509$
- $\sin 2.109 \approx 0.861$

Thus:

$$
z_1 \approx 1.603 \times (-0.509) + i \times 1.603 \times 0.861
$$

$$
z_1 \approx -0.815 + 1.382 i
$$

Root 3: $k = 2$

$$
z_2 = 1.603 \left(\cos \frac{0.245 + 4\pi}{3} + i \sin \frac{0.245 + 4\pi}{3} \right)
$$

$$
z_2 = 1.603 \left(\cos \frac{0.245 + 12.566}{3} \right) + i \sin \left(\frac{0.245 + 12.566}{3} \right)
$$

$$
z_2 = 1.603 \left(\cos 4.229 + i \sin 4.229 \right)
$$

Calculating:

- $\cos 4.229 \approx -0.491$
- $\sin 4.229 \approx -0.871$

Thus:

$$
z_2 \approx 1.603 \times (-0.491) + i \times 1.603 \times (-0.871)
$$

$$
z_2 \approx -0.787 - 1.396 i
$$

Step 4: Expressing Roots in \(a + bi \) Form

Now, summarizing the roots explicitly:

\[
\boxed{
\begin{aligned}
z_1 &\approx 1.597 + 0.131i \\
z_2 &\approx -0.815 + 1.382i \\
z_3 &\approx -0.787 - 1.396i
\end{aligned}
}
\]

These are approximate numerical representations, but they are sufficiently precise for most purposes.

Step 5: Verifying the Roots

While the approximate roots are helpful, verifying the roots explicitly involves calculating their cubes and confirming they equal \(4 + i \).

For illustrative purposes, let's verify the first root:

\[
z_1 \approx 1.597 + 0.131i
\]

Calculate \(z_1^3 \):

- First, find \(z_1^2 \):

\[
(1.597 + 0.131i)^2 = 1.597^2 + 2 \times 1.597 \times 0.131 i + (0.131 i)^2
\]
\[
= 2.551 + 0.418 i - 0

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