Find Steady-state Solution To The Problem U T +u Xx +u=0,u(x,0)=x 2 ,u(0,t)=2,u(/2,t)=1

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In this comprehensive guide, we will explore the process of finding the steady-state solution to the given partial differential equation (PDE) with specified initial and boundary conditions. Understanding such problems is fundamental in mathematical physics and engineering, especially when analyzing systems that tend toward equilibrium over time.

Understanding the Problem and Its Components

Before diving into the solution process, it is essential to understand the problem's structure and the significance of each component.

The PDE and Its Physical Interpretation

The PDE presented is:

$$
u_t + u_{xx} + u = 0
$$

where:
- $u(t, x)$ is the unknown function of time t and spatial coordinate x.
- u_t denotes the partial derivative of u with respect to time.
- u_{xx} is the second partial derivative of u with respect to space.

This PDE models phenomena such as heat conduction with reaction terms, diffusion processes with source/sink effects, or other physical systems where the change over time depends on diffusion and a linear term.

Initial and Boundary Conditions

The problem specifies:
- Initial condition:

$$
u(x, 0) = x^2
$$

- Boundary conditions:

$$
u(0, t) = 2 \quad \text{and} \quad u\left(\frac{1}{2}, t\right) = 1
$$

These conditions describe the state of the system at the initial time and the fixed values at the spatial boundaries over time.

Defining the Steady-State Solution

The steady-state solution $u_s(x)$ is the solution that the system approaches as $t \to \infty$. It satisfies the PDE when the time derivative is zero:
$$
u_t = 0
$$
Thus, the steady-state PDE reduces to an ordinary differential equation (ODE):
$$
u_{xx} + u = 0
$$

Our goal is to find $u_s(x)$ satisfying:
$$
u_{xx} + u = 0
$$
with boundary conditions:
$$
u(0) = 2, \quad u\left(\frac{1}{2}\right) = 1
$$

Note that the initial condition is no longer relevant for the steady-state, but it indicates the system's initial configuration.

Solving the Steady-State Equation

The steady-state ODE is:
$$
u_{xx} + u = 0
$$
which is a second-order linear homogeneous differential equation.

General Solution of the ODE

The characteristic equation associated with the ODE is:
$$
r^2 + 1 = 0
$$
solving for r:
$$
r = \pm i
$$
The general solution is:
$$
u_s(x) = A \cos x + B \sin x
$$
where A and B are constants to be determined by boundary conditions.

Applying Boundary Conditions

Using the boundary conditions:

$$
u(0) = 2 \Rightarrow u_s(0) = A \cos 0 + B \sin 0 = A \times 1 + B \times 0 = A = 2
$$

$$
u\left(\frac{1}{2}\right) = 1 \Rightarrow u_s\left(\frac{1}{2}\right) = 2 \cos \left(\frac{1}{2}\right) + B \sin \left(\frac{1}{2}\right) = 1
$$

Calculate known trigonometric values:

$$
\cos \left(\frac{1}{2}\right) \approx 0.8776
$$

$$
\sin \left(\frac{1}{2}\right) \approx 0.4794
$$

Substituting:

$$
2 \times 0.8776 + B \times 0.4794 = 1
$$

$$
1.7552 + 0.4794 B = 1
$$

Solve for B:

$$
0.4794 B = 1 - 1.7552 = -0.7552
$$

$$
B = \frac{-0.7552}{0.4794} \approx -1.577
$$

Resulting steady-state solution:

$$
u_s(x) = 2 \cos x - 1.577 \sin x
$$

Verification and Interpretation

The derived steady-state solution:

$$
u_s(x) = 2 \cos x - 1.577 \sin x
$$

satisfies the boundary conditions at $x=0$ and $x=\frac{1}{2}$. It represents the equilibrium state of the system.

- Physical Significance: The solution indicates how the system stabilizes, with the boundary conditions dictating the fixed values at the edges.
- Behavior: The sinusoidal form reflects oscillations within the domain, weighted by the boundary conditions.

Additional Considerations

While the steady-state solution provides critical insight, understanding the transient behavior (how the system approaches this equilibrium) involves solving the full PDE with the initial condition.

Transient Solution (Optional Overview)

To explore the transient dynamics, one would employ methods such as separation of variables, Fourier series, or Laplace transforms to solve the original PDE:

$$
u_t + u_{xx} + u = 0
$$

with the initial condition $u(x, 0) = x^2$ and boundary conditions $u(0, t) = 2$, $u(1/2, t) = 1$.

The general solution would be a superposition of the steady-state solution and a transient component that decays over time, reflecting the system's evolution toward equilibrium.

Summary of the Solution Process

To summarize, the steps to find the steady-state solution are:

1. Set $u_t = 0$ to reduce the PDE to an ODE: $u_{xx} + u = 0$.
2. Find the general solution: $u_s(x) = A \cos x + B \sin x$.
3. Apply boundary conditions to determine constants A and B:
   - $u(0) = 2 \Rightarrow A=2$.
   - $u(1/2) = 1 \Rightarrow 2 \cos (1/2) + B \sin (1/2) = 1$.
4. Solve for B, obtaining approximately $B \approx -1.577$.
5. Write the final steady-state solution explicitly:
   $$
   u_s(x) = 2 \cos x - 1.577 \sin x
   $$

Conclusion

In conclusion, the steady-state solution to the PDE $u_t + u_{xx} + u = 0$ with the specified boundary conditions is a harmonic function characterized by sinusoidal components that satisfy the boundary constraints. This solution provides vital insights into the long-term behavior of the system and serves as a foundation for further analysis, including transient dynamics and stability considerations.

Understanding how to derive and interpret such solutions is crucial for applied mathematicians, engineers, and physicists working with diffusion-reaction systems, thermal conduction problems, and other related phenomena.

Frequently Asked Questions

What is the steady-state solution for the given PDE with boundary conditions?

The steady-state solution is found by setting the time derivative to zero, leading to the ODE $u'' + u = 0$, with boundary conditions $u(0) = 2$ and $u(1/2) = 1$. Solving yields $u_s(x) = A \cos x + B \sin x$, with constants determined by the boundary conditions.

How do we derive the steady-state solution from the PDE u_T + u u_xx + u = 0?

In steady-state, $u_T=0$, simplifying the PDE to $u u_{xx} + u = 0$. Recognizing this as an ordinary differential equation, we solve for $u(x)$, applying boundary conditions to find the specific solution.

What is the physical interpretation of the steady-state solution in this problem?

The steady-state solution represents the long-term behavior of the system where the solution no longer changes with time, indicating equilibrium conditions governed by the balance dictated by the PDE and boundary conditions.

How do boundary conditions u(0,t)=2 and u(1/2,t)=1 influence the steady-state solution?

They determine the constants in the general solution of the ODE derived from the PDE, ensuring the solution matches the specified values at the boundaries, leading to a unique steady-state profile.

Can the method of separation of variables be used to find the steady-state solution? Why or why not?

Separation of variables is typically used for time-dependent problems. For the steady-state, which involves solving an ODE, direct integration methods are more appropriate. However, considering the PDE as a boundary value problem for $u(x)$ is standard here.

What are the steps involved in solving for the steady-state solution of this PDE?

First, set $u_T=0$ to obtain an ODE, then solve the ODE $u u'' + u=0$ or its equivalent, apply boundary conditions $u(0)=2$ and $u(1/2)=1$ to find the constants, and write the explicit solution.

Is the steady-state solution unique for this problem?

Yes, given the boundary conditions are specified at two points and the ODE is linear, the solution is unique.

What challenges might arise when solving for the steady-state solution in this problem?

Nonlinearities in the PDE, boundary conditions at two points, and ensuring the solution satisfies the boundary conditions can pose challenges. Analytical solutions may involve solving transcendental equations for constants.

How does the initial condition u(x,0)=x^2 relate to the steady-state solution?

The initial condition describes the system's state at t=0; the steady-state solution represents the long-term behavior after evolution over time. The initial condition influences transient solutions but not the steady-state directly.

What is the significance of finding the steady-state solution in this PDE problem?

It helps understand the equilibrium state of the system, which is important for predicting long-term behavior and stability without solving the full time-dependent problem.

Additional Resources

Find Steady-state Solution To The Problem U T + u_xx + u = 0, u(x,0) = x^2, u(0,t) = 2, u(1/2,t) = 1

In the realm of mathematical physics and applied mathematics, partial differential equations (PDEs) serve as fundamental tools for modeling various phenomena—ranging from heat conduction to wave propagation. One such PDE, the heat equation with an additional reaction term, appears frequently in studies of temperature distribution and diffusion processes with sources or sinks. Today, we explore a specific problem: finding the steady-state solution to the PDE

$$U_T + u_{xx} + u = 0$$

subject to initial and boundary conditions:

- Initial condition: $u(x, 0) = x^2$,
- Boundary conditions: $u(0, t) = 2$ and $u(1/2, t) = 1$.

Our goal is to understand the behavior of the system as time progresses and to find the steady-state solution—i.e., the solution that remains unchanged over time. This exploration not only demonstrates the power of analytical methods in PDEs but also illustrates how boundary and initial conditions influence system behavior.

Understanding the Problem: PDEs and Boundary Value Conditions

Before diving into the solution process, it's essential to clarify the problem's physical and mathematical context.

The PDE: Heat with Reaction Term

The equation

$$U_T + u_{xx} + u = 0$$

can be viewed as a heat equation with a reaction term u. Here:

- U_T is the partial derivative of u with respect to time t,
- u_{xx} is the second spatial derivative with respect to x,
- The term $+ u$ acts as a reaction or source/sink term, influencing how the temperature evolves.

This PDE models scenarios where heat conduction is affected by reactions or sources, such as chemical processes or biological systems.

Boundary and Initial Conditions

- Initial condition: $u(x, 0) = x^2$, which sets the initial temperature distribution along the spatial domain $x \in [0, 1/2]$.
- Boundary conditions:
- At $x = 0$: $u(0, t) = 2$,
- At $x = 1/2$: $u(1/2, t) = 1$.

These conditions specify fixed temperatures at the boundaries, influencing how heat flows within the domain.

Steady-State Solution: Concept and Significance

The steady-state solution corresponds to the temperature distribution that remains constant over time. Mathematically, this is achieved when

$$U_T = 0,$$

reducing the PDE to an ordinary differential equation (ODE):

$$u_{xx} + u = 0.$$

Finding this solution involves solving the ODE with the given boundary conditions. Physically, it describes the equilibrium temperature profile after sufficient time has elapsed, where the system's transient behaviors have decayed.

Step 1: Deriving the Steady-State Equation

Set $U_T = 0$ to focus on the steady-state:

$$
u_{xx} + u = 0,
$$

which is a second-order linear homogeneous ODE with constant coefficients.

Step 2: Solving the ODE

The standard approach involves solving the characteristic equation:

$$
r^2 + 1 = 0,
$$

which yields complex roots:

$$
r = \pm i.
$$

Thus, the general solution to the differential equation is:

$$
u_{ss}(x) = A \cos x + B \sin x,
$$

where A and B are constants to be determined from boundary conditions.

Step 3: Applying Boundary Conditions

Recall the boundary conditions:

- $u(0) = 2$,
- $u(1/2) = 1$.

Applying $u(0) = 2$:

$$
u_{ss}(0) = A \cos 0 + B \sin 0 = A \times 1 + B \times 0 = A = 2.
$$

Applying $u(1/2) = 1$:

$$
u_{ss}(1/2) = 2 \cos (1/2) + B \sin (1/2) = 1.
$$

Calculate $\cos(1/2)$ and $\sin(1/2)$:

$$
\cos(1/2) \approx 0.8776, \quad \sin(1/2) \approx 0.4794.
$$

Substitute these values:

$$
2 \times 0.8776 + B \times 0.4794 = 1,
$$
$$
1.7552 + 0.4794 B = 1,
$$
$$
0.4794 B = 1 - 1.7552 = -0.7552,
$$
$$
B = \frac{-0.7552}{0.4794} \approx -1.577.
$$

Thus, the steady-state solution is:

$$
u_{ss}(x) = 2 \cos x - 1.577 \sin x.
$$

This function describes the equilibrium temperature distribution in the system.

Step 4: Interpreting the Steady-State Solution

The derived solution

$$
u_{ss}(x) = 2 \cos x - 1.577 \sin x,
$$

provides crucial insights:

- It satisfies the boundary conditions exactly.
- It is independent of initial conditions, reflecting the long-term behavior after transients decay.
- The form combines sinusoidal functions, characteristic of solutions to second-order linear ODEs with constant coefficients.

Step 5: Addressing the Initial Condition

While the steady-state solution describes the long-term equilibrium, it does not satisfy the initial condition $u(x, 0) = x^2$. The initial condition influences the transient part of the solution, which decays over time due to the presence of the reaction term u in the PDE.

In practice, the full solution is composed of:

$$
u(x, t) = u_{ss}(x) + u_{transient}(x, t),
$$

$$

where $u_{transient}$ satisfies the homogeneous PDE with zero boundary conditions and initial data:

$$
u_{transient}(x, 0) = u(x, 0) - u_{ss}(x) = x^2 - \left(2 \cos x - 1.577 \sin x \right).
$$

This transient component diminishes over time, leaving only $u_{ss}(x)$ as $t \to \infty$.

Advanced Considerations: Transients and Complete Solution

Although the steady-state solution gives the long-term equilibrium, understanding the transient behavior requires solving the PDE for $u_{transient}$. This involves:

- Applying separation of variables,
- Expanding the initial condition in eigenfunctions,
- Solving the resulting eigenvalue problem.

However, for the purpose of identifying the steady state, the derived $u_{ss}(x)$ suffices.

Practical Implications and Applications

The process outlined here exemplifies how boundary conditions shape the ultimate temperature profile in physical systems:

- Engineering: Design of heat exchangers and temperature regulation systems.
- Environmental Science: Modeling temperature distributions in bodies of water or soil.
- Biology: Diffusion of nutrients or chemicals within tissues.

Knowing the steady-state solution helps engineers and scientists predict the final system configuration without solving the entire transient dynamics.

Summary: Key Takeaways

- The steady-state solution to the PDE $u_{xx} + u = 0$ with boundary conditions $u(0) = 2$ and $u(1/2) = 1$ is:

$$
\boxed{
u_{ss}(x) = 2 \cos x - 1.577 \sin x.
}
$$

- This solution satisfies the boundary conditions exactly and describes the equilibrium temperature distribution.
- The initial condition influences the transient behavior, which decays over

time, leaving the steady state as the long-term outcome.
- Understanding the steady state is crucial in designing systems that require stable temperature profiles or concentration distributions.

Concluding Remarks

Finding the steady-state solution of a PDE is a central task in applied mathematics, bridging theoretical analysis with real-world applications. By converting the PDE into an ordinary differential equation and applying boundary conditions, we obtain solutions that reveal the system's long-term behavior. The example discussed illustrates the elegance and power of classical methods, offering insights into complex systems governed by differential equations.

Whether in engineering, physics, or biological sciences, mastering these techniques enables practitioners to predict, control, and optimize systems across diverse domains.

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