Find The Area Of The Region Enclosed By One Loop Of The Curve. R = 2 4 Sin() (inner Loop)

Find The Area Of The Region Enclosed By One Loop Of The Curve: R = 2 4 Sin(θ) (Inner Loop)

The problem of finding the area enclosed by a specific loop of a polar curve is a classical topic in calculus, especially in the study of curves expressed in polar coordinates. The given curve, R = 2 4 Sin(θ), suggests a sinusoidal relationship between the radius and the angle, which naturally leads to a curve with multiple loops. Our goal is to determine the area enclosed by one of these loops, specifically the inner loop, which requires understanding the curve's behavior, identifying the interval of θ that corresponds to this loop, and then applying the formula for polar areas.

Understanding the Curve R = 2 4 Sin(θ)

Interpreting the Equation

At first glance, the curve R = 2 4 Sin(θ) appears to be a scaled sinusoid in polar form. To make sense of it, consider rewriting the equation as:

- R = (2)(4) Sin(θ) = 8 Sin(θ)

This form indicates that the radius varies sinusoidally with θ, with a maximum radius of 8 when Sin(θ) = 1, i.e., at θ = 90°, and a minimum radius of -8 when Sin(θ) = -1, at θ = 270°. Since negative radii in polar coordinates indicate a point in the opposite direction, the curve will have symmetry and loops corresponding to these extrema.

Graphical Behavior of R = 8 Sin(θ)

The curve is a lemniscate-like shape with the following characteristics:

1. Maximum radius of 8 at θ = 90° (π/2 radians).
2. Zero radius at θ = 0 and θ = π (0° and 180°); these points correspond to the curve crossing the origin.

3. Negative radius values imply the curve extends in the opposite direction, effectively creating a symmetric inner loop.

To identify the inner loop, we analyze where the radius becomes negative and the curve traces back over the origin, creating a distinct loop enclosed within the entire shape.

Identifying the Limits of Integration for the Inner Loop

Finding the θ-values Corresponding to the Inner Loop

For $R = 8 \sin(\theta)$, the inner loop occurs where the radius becomes negative, i.e., when $\sin(\theta) < 0$. Specifically, the inner loop is traced out in the interval where the curve dips below the origin, corresponding to θ in:

- $\pi < \theta < 2\pi$

However, more precisely, the inner loop is the region where the radius completes a full loop below the origin, which occurs between specific θ-values where $R = 0$, i.e., where $\sin(\theta) = 0$.

Finding θ where R = 0

Set $R = 0$:

$8 \sin(\theta) = 0$

Thus, $\sin(\theta) = 0$, which implies:

- $\theta = 0, \pi, 2\pi, 3\pi, ...$

Within the principal interval $[0, 2\pi]$, the curve crosses the origin at $\theta = 0$, π, and 2π. The inner loop appears between $\theta = \pi$ and $\theta = 2\pi$, where the radius is negative, and the curve traces out the inner loop in this interval.

Calculating the Area of the Inner Loop

The Polar Area Formula

The area A enclosed by a polar curve $R(\theta)$ between $\theta = a$ and $\theta = b$ is given by:

$$A = (1/2) \int_a^b [R(\theta)]^2 \, d\theta$$

For the inner loop, we consider θ from π to 2π, where the curve traces the inner loop.

Setting Up the Integral

Since $R = 8 \sin(\theta)$, then:

$$A_{inner} = (1/2) \int_{\theta=\pi}^{\theta=2\pi} [8 \sin(\theta)]^2 \, d\theta = (1/2) \int_{\pi}^{2\pi} 64 \sin^2(\theta) \, d\theta$$

Factor out constants:

$$A_{inner} = (1/2) \; 64 \int_{\pi}^{2\pi} \sin^2(\theta) \, d\theta = 32 \int_{\pi}^{2\pi} \sin^2(\theta) \, d\theta$$

Evaluating the Integral $\int \sin^2(\theta) \, d\theta$

Recall the trigonometric identity:

$$\sin^2(\theta) = (1 - \cos(2\theta))/2$$

Substitute into the integral:

$$A_{inner} = 32 \int_{\pi}^{2\pi} (1 - \cos(2\theta))/2 \, d\theta = 16 \int_{\pi}^{2\pi} (1 - \cos(2\theta)) \, d\theta$$

Calculating the Integral

1. Integral of 1 with respect to θ:

   $$\int_{\pi}^{2\pi} 1 \, d\theta = (2\pi - \pi) = \pi$$

2. Integral of $\cos(2\theta)$ with respect to θ:

   $$\int \cos(2\theta) \, d\theta = (1/2) \sin(2\theta)$$

Putting it all together:

$A_{inner} = 16 [\int_{\pi}^{2\pi} 1 \, d\theta - \int_{\pi}^{2\pi} \cos(2\theta) \, d\theta] = 16 [\pi - (1/2) \sin(2\theta) |_{\pi}^{2\pi}]$

Evaluating the Limits

Calculate $\sin(2\theta)$ at the bounds:

At $\theta = 2\pi$: $\sin(4\pi) = 0$
At $\theta = \pi$: $\sin(2\pi) = 0$

Thus:

$A_{inner} = 16 [\pi - (1/2)(0 - 0)] = 16 \pi$

Final Result: Area of the Inner Loop

The area enclosed by one loop of the curve $R = 8 \sin(\theta)$, corresponding to the inner loop between $\theta = \pi$ and $\theta = 2\pi$, is:

$A_{inner} = 16\pi$

Thus, the area of the inner loop is 16π square units.

Summary and Additional Insights

- The curve $R = 8 \sin(\theta)$ creates multiple loops, with the inner loop traced out from $\theta = \pi$ to $\theta = 2\pi$.
- The area enclosed by this inner loop is found using the polar area formula, integrating R^2 over the relevant interval.
- Using the identity for $\sin^2(\theta)$, the integral simplifies, leading to a clean expression for the area.
- The final area is 16π, emphasizing the symmetry and periodicity inherent in sinusoidal polar curves.

Conclusion

Calculating the area of a particular loop in a polar curve involves understanding the behavior of the curve, identifying the correct limits of integration, and applying the polar area formula with appropriate trigonometric identities. For the curve $R = 8 \sin(\theta)$, the inner loop's area is 16π square units, a result that elegantly combines geometry, calculus, and trigonometry. This approach can be extended to analyze more complex polar curves with multiple loops and symmetries, providing a robust framework for exploring the geometric properties of polar equations.

Frequently Asked Questions

What is the general method to find the area enclosed by one loop of the polar curve R = 2 - 4 sin θ?

The area enclosed by one loop of a polar curve $R(\theta)$ can be found using the integral $A = (1/2) \int_{\theta 1}^{\theta 2} R(\theta)^2 \, d\theta$, where $\theta 1$ and $\theta 2$ are the limits that correspond to one complete loop.

For the curve R = 2 - 4 sin θ, how do we determine the limits of integration for finding the inner loop's area?

The limits are found by solving $R = 0$: $0 = 2 - 4 \sin \theta \Rightarrow \sin \theta = 0.5 \Rightarrow \theta = \pi/6$ and $5\pi/6$. These angles mark where the curve intersects the origin, defining the bounds of the inner loop.

What is the value of θ at the start and end of the inner loop for R = 2 - 4 sin θ?

The inner loop is enclosed between $\theta = \pi/6$ and $\theta = 5\pi/6$, where the curve crosses the origin, completing one loop.

How do you set up the integral to find the area of the inner loop of R = 2 - 4 sin θ?

The area is given by $A = (1/2) \int_{\pi/6}^{5\pi/6} [2 - 4 \sin \theta]^2 \, d\theta$.

What is the simplified form of the integrand when calculating the area of the loop for R = 2 - 4 sin θ?

Expanding the square, the integrand becomes $(2 - 4 \sin \theta)^2 = 4 - 16 \sin \theta + 16 \sin^2 \theta$.

What steps are involved in evaluating the integral

for the area enclosed by one loop of R = 2 - 4 sin θ?

First, expand the integrand, then integrate term-by-term: ∫ 4 dθ, ∫ -16 sin θ dθ, and ∫ 16 sin^2 θ dθ, applying standard integration techniques and limits.

How do you handle the integral of sin^2 θ when calculating the area?

Use the identity sin^2 θ = (1 - cos 2θ)/2 to rewrite and integrate easily: ∫ sin^2 θ dθ = (θ/2) - (sin 2θ)/4 + C.

What is the final expression for the area of the inner loop of R = 2 - 4 sin θ?

After evaluating the integral, the area A = (1/2) [integrated result between θ = π/6 and 5π/6], which simplifies to A = 8π - 8.

Why is it important to identify the correct limits when calculating the area enclosed by a loop of a polar curve?

Correct limits ensure that only the region corresponding to one loop is considered, avoiding overcounting or missing parts of the area, which is essential for an accurate calculation.

Additional Resources

Find The Area Of The Region Enclosed By One Loop Of The Curve R = 2 + 4 sin(θ) (inner Loop)

Understanding how to find the area enclosed by a single loop of a polar curve is a fundamental skill in calculus, especially in the study of curves expressed in polar coordinates. The specific problem of finding the area of the region enclosed by one loop of the curve R = 2 + 4 sin(θ) (inner Loop) involves analyzing the properties of the curve, identifying key points, and applying the appropriate integral formulas. In this guide, we will walk through each step in detail, providing a comprehensive approach to solving this classic problem.

Introduction to the Curve: R = 2 + 4 sin(θ)

Before diving into the calculations, it's essential to understand the nature of the curve described by the polar equation:

$$ R = 2 + 4 \sin(\theta) $$

This is a type of limacon, a family of polar curves characterized by their distinctive loops and dimpled shapes depending on the parameters involved.

Key Characteristics:

- The curve is symmetric with respect to the vertical axis (the line $\theta = 0$).
- When the coefficient of $\sin(\theta)$ is greater than the constant term, the limacon exhibits an inner loop.
- In this case, since $4 > 2$, the curve has an inner loop, which is the focus of our area calculation.

Visualizing the Curve:

- At $\theta = 0$: $R = 2 + 4\ 0 = 2$
- At $\theta = \pi/2$: $R = 2 + 4\ 1 = 6$
- At $\theta = \pi$: $R = 2 + 4\ 0 = 2$
- At $\theta = 3\pi/2$: $R = 2 + 4\ (-1) = -2$

Note: Negative R values indicate the point is plotted in the opposite direction, effectively flipping the point across the origin, which is crucial for understanding the inner loop.

Step 1: Identifying the Inner Loop

To find the area enclosed by the inner loop, we need to determine the range of θ where the curve traces out this loop.

When does the inner loop occur?

- The inner loop appears where $R(\theta) < 0$, because in polar coordinates, negative R values correspond to points plotted in the opposite direction, creating the loop.
- Set $R = 0$ and solve for θ:

$$ 0 = 2 + 4 \sin(\theta) $$

$$ 4 \sin(\theta) = -2 $$

$$ \sin(\theta) = -\frac{1}{2} $$

- The solutions are:

$$ \theta = \frac{7\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6} $$

These are the angles where the curve intersects the origin, marking the bounds of the inner loop.

Range of θ for the inner loop

- The inner loop is traced out as θ varies between:

$$ \boxed{\theta = \frac{7\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6}} $$

Within this interval, $R < 0$, and the curve forms the inner loop.

Step 2: Setting Up the Area Integral

The area A enclosed by a polar curve between $\theta = a$ and $\theta = b$ is given by:

$$
A = \frac{1}{2} \int_{a}^{b} R^2 \, d\theta
$$

Applying to Our Curve:

- Since the inner loop is traced between $\theta = 7\pi/6$ and $11\pi/6$, the area of the inner loop is:

$$
A_{\text{inner}} = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} [R(\theta)]^2 \, d\theta
$$

- Substitute $R(\theta) = 2 + 4 \sin(\theta)$:

$$
A_{\text{inner}} = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} [2 + 4 \sin(\theta)]^2 \, d\theta
$$

Step 3: Simplify the Integral

Expand the squared term:

$$
[2 + 4 \sin(\theta)]^2 = 4 + 16 \sin(\theta) + 16 \sin^2(\theta)
$$

Thus,

$$
A_{\text{inner}} = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} \left(4 + 16 \sin(\theta) + 16 \sin^2(\theta) \right) d\theta
$$

Break into separate integrals:

$$
A_{\text{inner}} = \frac{1}{2} \left[\int_{7π/6}^{11π/6} 4 \, dθ + \int_{7π/6}^{11π/6} 16 \sin(θ) \, dθ + \int_{7π/6}^{11π/6} 16 \sin^2(θ) \, dθ \right]
$$

Step 4: Evaluate Each Integral

1. Integral of a constant:

$$
\int_{7π/6}^{11π/6} 4 \, dθ = 4 \times \left(\frac{11π}{6} - \frac{7π}{6} \right) = 4 \times \frac{4π}{6} = 4 \times \frac{2π}{3} = \frac{8π}{3}
$$

2. Integral of sine:

$$
\int_{7π/6}^{11π/6} 16 \sin(θ) \, dθ = -16 \cos(θ) \bigg|_{7π/6}^{11π/6}
$$

Calculate:

$$
-16 [\cos(11π/6) - \cos(7π/6)]
$$

Recall:

- $\cos(11π/6) = \cos(2π - π/6) = \cos(π/6) = \frac{\sqrt{3}}{2}$
- $\cos(7π/6) = \cos(π + π/6) = -\cos(π/6) = -\frac{\sqrt{3}}{2}$

Thus,

$$
-16 \left(\frac{\sqrt{3}}{2} - \left(- \frac{\sqrt{3}}{2} \right) \right) = -16 \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right) = -16 \times \sqrt{3}
$$

Simplify:

$$
-16 \times \sqrt{3}
$$

3. Integral of $\sin^2(θ)$:

Use the power-reduction identity:

\[
\sin^2(θ) = \frac{1 - \cos(2θ)}{2}
\]

Therefore,

\[
\int_{7π/6}^{11π/6} 16 \sin^2(θ) \, dθ = 16 \times \frac{1}{2}
\int_{7π/6}^{11π/6} (1 - \cos(2θ)) \, dθ
\]

\[
= 8 \left[\int_{7π/6}^{11π/6} 1 \, dθ - \int_{7π/6}^{11π/6} \cos(2θ) \, dθ
\right]
\]

Calculate separately:

- \(\int_{7π/6}^{11π/6} 1 \, dθ = \frac{4π}{6} = \frac{2π}{3}\)

- \(\int_{7π/6}^{11π/6} \cos(2θ) \, dθ = \frac{1}{2} \sin(2θ)
\bigg|_{7π/6}^{11π/6}\)

Evaluate:

\[
\frac{1}{2} [\sin(2 \times \frac{11π}{6}) - \sin(2 \times \frac{7π}{6})] =
\frac{1}{2} [\sin(\frac{22π}{6}) - \sin(\frac{14π}{6})]
\]

Simplify angles:

- \(\sin(\frac{22π}{6}) = \sin(\frac{11π}{3})\)

Since \(\sin(11π/3) = \sin(11π/3 - 2π \times 3) = \sin(11π/3 - 6π) =
\sin(11π/3 - 18π/3) = \sin(-7π/3)\)

And \(\sin(-x) = -\sin(x)\), so:

\[
\sin(-7π/3) = -\sin(7π/3)
\]

Now, \(\sin(7π/3) = \sin(2π + π/3) = \sin(π/3)

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