Find The Average Value Of F Over The Given Rectangle. F(x, Y) = 4ey X+ey , R = [0, 6] [0, 1]

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Understanding how to find the average value of a function over a specified region is a fundamental concept in calculus, especially in the study of multivariable functions. When dealing with functions of two variables, such as F(x, y), the process involves integrating the function over the given region and then dividing by the area of that region. In this article, we will explore how to compute the average value of the function $F(x, y) = 4e^{y} e^{x}$ over the rectangle $R = [0, 6] \times [0, 1]$.

Introduction to the Problem

Calculating the average value of a function over a region is essential in various fields, including physics, engineering, and economics. It allows us to understand the "mean" behavior of the function within a specific domain. For the given function and region, the goal is to determine:

- The double integral of F(x, y) over R
- The area of R
- The average value, which is the ratio of the integral to the area

Let's define the problem more precisely:

- Function: $F(x, y) = 4e^{y} e^{x}$
- Region: $R = [0, 6] \times [0, 1]$

This setup provides a rectangular region in the xy-plane, spanning from x=0 to x=6 and y=0 to y=1.

Understanding the Concept of Average Value of a Function Over a Region

The average value of a function F(x, y) over a region R is given by the formula:

$$
\text{Average value} = \frac{1}{\text{Area of } R} \iint_{R} F(x, y) \, dA
$$

Where:

- $\iint_{R} F(x, y) \, dA$ is the double integral of F over R
- dA represents an infinitesimal area element, which in Cartesian coordinates is $dx\,dy$

This calculation involves two main steps:

1. Compute the double integral of the function over the specified region
2. Determine the area of the region

The ratio of these two quantities gives the average value of the function over the region.

Step-by-Step Solution

Let's proceed step-by-step to calculate the average value of F(x, y) over R.

Step 1: Express the Double Integral

Given the rectangular region R = [0, 6] × [0, 1], the double integral becomes:

$$
\iint_{R} F(x, y) \, dA = \int_{x=0}^{6} \int_{y=0}^{1} 4 e^{y} e^{x} \, dy\, dx
$$

Since F(x, y) can be separated into the product of functions of x and y, it simplifies the integral:

$$
F(x, y) = 4 e^{x} e^{y}
$$

Thus, the double integral is:

$$
\int_{x=0}^{6} \int_{y=0}^{1} 4 e^{x} e^{y} \, dy\, dx
$$

Step 2: Integrate with Respect to y

First, perform the inner integral over y:

$$
\int_{0}^{1} 4 e^{x} e^{y} \, dy
$$

Since e^{x} is a constant with respect to y:

$$
4 e^{x} \int_{0}^{1} e^{y} \, dy
$$

The integral of e^{y} from 0 to 1 is:

$$
\int_{0}^{1} e^{y} \, dy = e^{y} \big|_{0}^{1} = e^{1} - e^{0} = e - 1
$$

Therefore, the inner integral becomes:

$$
4 e^{x} (e - 1)
$$

Step 3: Integrate with Respect to x

Now, the outer integral is:

$$
\int_{0}^{6} 4 e^{x} (e - 1) \, dx
$$

Constants can be factored out:

$$
4 (e - 1) \int_{0}^{6} e^{x} \, dx
$$

The integral of e^{x} from 0 to 6 is:

$$
e^{x} \big|_{0}^{6} = e^{6} - e^{0} = e^{6} - 1
$$

Thus, the double integral evaluates to:

$$
4 (e - 1) (e^{6} - 1)
$$

Step 4: Find the Area of R

The area of the rectangle R = [0, 6] × [0, 1] is simply:

$$
\text{Area} = \text{length} \times \text{width} = (6 - 0) \times (1 - 0) = 6 \times 1 = 6
$$

Step 5: Compute the Average Value

Finally, the average value of F over R is:

$$
\boxed{
\text{Average value} = \frac{1}{\text{Area}} \times \text{Double integral} = \frac{1}{6} \times 4 (e - 1) (e^{6} - 1)
}
$$

Simplifying:

$$
\text{Average value} = \frac{2}{3} (e - 1) (e^{6} - 1)
$$

This is the exact expression for the average value of the function F over the rectangular region R.

Interpretation and Significance

Understanding the Result

The calculated average value provides insight into the "mean" behavior of the function across the specified region. Since F(x, y) involves exponential functions, the average value captures the combined growth behavior across the rectangle:

- As e^{6} is significantly large, the average value reflects the

exponential increase in the function's magnitude over the region
- The factors $(e - 1)$ and $(e^{6} - 1)$ emphasize the exponential growth in y and x directions, respectively

Practical Applications

Knowing how to compute the average value of multivariable functions is useful in:

- Physics: Calculating average heat distribution over a surface
- Economics: Estimating average profit or cost over a range of variables
- Engineering: Assessing average stress or strain over a component

Visualizing the Function

Graphically, $F(x, y)$ increases exponentially in both x and y directions. The average value effectively summarizes the overall "central tendency" of the function within the specified rectangle.

Additional Considerations

Variations in Region

While this example considers a simple rectangular region, the process generalizes to more complex regions:

- For irregular regions, the limits of integration change accordingly
- For curved or non-rectangular regions, techniques like changing variables or parameterization may be necessary

Numerical Approximations

In cases where the integral cannot be computed analytically, numerical methods such as Simpson's rule, Gaussian quadrature, or Monte Carlo integration can approximate the average value.

Software Tools

Mathematical software like Wolfram Mathematica, MATLAB, or online integral calculators can assist in computing complex integrals efficiently.

Summary

Calculating the average value of a function over a region involves integrating the function over that region and dividing by the region's area. For the specific function $F(x, y) = 4 e^{x} e^{y}$ over the rectangle $R = [0, 6] \times [0, 1]$:

- The double integral evaluates to $4 (e - 1) (e^{6} - 1)$
- The area of R is 6
- The average value is $\frac{2}{3} (e - 1) (e^{6} - 1)$

This process exemplifies the fundamental techniques in multivariable calculus for averaging functions over regions, with broad applications across scientific and engineering disciplines.

Keywords: average value of a function, double integral, multivariable

calculus, exponential functions, region integration, rectangle, mathematical analysis, calculus applications

Frequently Asked Questions

What is the goal when finding the average value of the function $F(x, y) = 4e^{x} + e^{y}$ over the rectangle $R = [0, 6] \times [0, 1]$?

The goal is to compute the average value of the function F over the specified rectangle, which involves integrating F over R and dividing by the area of R.

How do you set up the double integral to find the average value of $F(x, y) = 4e^{x} + e^{y}$ over $R = [0, 6] \times [0, 1]$?

The average value is given by (1/area of R) times the double integral of F over R. So, set up: $(1/\ (6\ 1)) \int_0^1 \int_0^6 (4e^{x} + e^{y})\, dx\, dy$.

What is the area of the rectangle $R = [0, 6] \times [0, 1]$?

The area is the length times width, so area = 6 1 = 6.

How do you evaluate the double integral of $F(x, y) = 4e^{x} + e^{y}$ over R?

Since the function is separable, integrate with respect to x and y separately: $\int_0^1 \int_0^6 (4e^{x} + e^{y})\, dx\, dy = \int_0^1 [\int_0^6 4e^{x}\, dx + \int_0^6 e^{y}\, dx]\, dy$.

What are the integrals of $4e^{x}$ and e^{y} with respect to x and y respectively?

$\int_0^6 4e^{x}\, dx = 4[e^{x}]_0^6 = 4(e^{6} - 1)$, and $\int_0^1 e^{y}\, dy = [e^{y}]_0^1 = e - 1$.

How do you compute the double integral of $F(x, y)$ over R?

Since the integrals are independent, the double integral becomes: $\int_0^1 [4(e^{6} - 1) + (e - 1)]\, dy = [4(e^{6} - 1) + (e - 1)]\ 1$, because the integration over y is straightforward.

What is the final step to find the average value of F over R?

Divide the value of the double integral by the area of R (which is 6): Average value = $[4(e^{6} - 1) + (e - 1)] / 6$.

What is the approximate numerical value of the average F over the rectangle?

Calculating: $e^{6} \approx 403.429$, so $4(e^{6} - 1) \approx 4(403.429 - 1) \approx 4\ 402.429 \approx 1609.716$. Then, sum: $1609.716 + (e - 1) \approx 1609.716 + (2.718 - 1) \approx 1609.716 + 1.718 \approx 1611.434$. Divide by 6: approximately 268.57.

Additional Resources

Find The Average Value Of F Over The Given Rectangle. $F(x, y) = 4e^{xy}$, $R = [0, 6] \times [0, 1]$

Introduction: Deciphering the Concept of Average Value in Multivariable Calculus

In the world of calculus, especially when dealing with functions of multiple variables, understanding the average value of a function over a specific region is a fundamental concept that bridges the gap between local behavior and global insights. This idea is particularly useful in fields such as physics, engineering, and economics, where understanding the overall behavior of a system across an area or volume can inform decision-making and analysis.

In this article, we explore how to find the average value of a given function, $F(x, y) = 4e^{xy}$, over a rectangular region $R = [0, 6] \times [0, 1]$. We will walk through the mathematical process step-by-step, from understanding the definition to executing the calculations, all while maintaining clarity and precision to ensure the concepts are accessible and meaningful.

Understanding the Concept of the Average Value of a Function

Before diving into the calculations, it’s crucial to understand what the "average value" of a function over a region entails.

Definition

The average value of a function $F(x, y)$ over a rectangular region R is given by the formula:

$$
\text{Average value} = \frac{1}{\text{Area of } R} \iint_{R} F(x, y) \, dA
$$

where:

- $\iint_{R} F(x, y) \, dA$ is the double integral of F over R,
- $\text{Area of } R$ is the measure of the region, calculated as the product of the lengths of its sides.

Intuitive Explanation

Think of the average value as the average height of a surface $z = F(x, y)$ when projected onto a flat plane over the specified region. This is akin to averaging the temperature across a rectangular area, where the temperature at each point influences the overall average.

Step 1: Clarifying the Region R

The given region $R = [0, 6] \times [0, 1]$ is a rectangle in the xy-plane.

- Along the x-axis, x varies from 0 to 6.
- Along the y-axis, y varies from 0 to 1.

The dimensions are straightforward:

- Width in the x-direction: $6 - 0 = 6$
- Height in the y-direction: $1 - 0 = 1$

Thus, the area of R is:

$$
\text{Area} = 6 \times 1 = 6
$$

Step 2: Setting Up the Double Integral

The next step involves expressing the average value mathematically:

$$
\text{Average} = \frac{1}{6} \iint_{R} 4 e^{xy} \, dA
$$

Since the region R is a rectangle, we can evaluate the double integral as an iterated integral:

$$
\iint_{R} F(x, y) \, dA = \int_{x=0}^{6} \int_{y=0}^{1} 4 e^{xy} \, dy \, dx
$$

This approach simplifies the calculation by integrating with respect to y first, then with respect to x.

Step 3: Computing the Inner Integral

Focus on the inner integral:

$$
I(x) = \int_{0}^{1} 4 e^{xy} \, dy
$$

Since x is treated as a constant during this integration, the integral becomes:

$$
I(x) = 4 \int_{0}^{1} e^{xy} \, dy
$$

\]

Calculating $\int e^{xy} \, dy$

The integral of an exponential function with a linear exponent is:

$$
\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C
$$

In our case, $a = x$, which is a constant with respect to y. Therefore:

$$
\int e^{xy} \, dy = \frac{1}{x} e^{xy} + C \quad \text{for } x \neq 0
$$

Handling the case when $x = 0$

When $x = 0$:

$$
I(0) = 4 \int_{0}^{1} e^{0 \cdot y} \, dy = 4 \int_{0}^{1} 1 \, dy = 4 \times (1 - 0) = 4
$$

Final expression for the inner integral

For $x \neq 0$:

$$
I(x) = 4 \times \left[\frac{1}{x} e^{xy} \right]_0^{1} = 4 \times \left(\frac{1}{x} e^{x \times 1} - \frac{1}{x} e^{x \times 0} \right)
$$

which simplifies to:

$$
I(x) = 4 \times \left(\frac{e^{x} - 1}{x} \right)
$$

and the special case at $x=0$ yields $I(0) = 4$, which matches the limit of the expression as $x \to 0$.

Step 4: Computing the Outer Integral

Now, the double integral becomes:

$$
\int_{0}^{6} I(x) \, dx = \int_{0}^{6} 4 \times \frac{e^{x} - 1}{x} \, dx
$$

This integral is more challenging because of the $\frac{e^{x} - 1}{x}$ term. However, for the purpose of finding the average value, we can write the expression explicitly and evaluate numerically or through approximation methods if needed.

Expressing the integral:

$$
\iint_{R} F(x, y) \, dA = \int_{0}^{6} 4 \times \frac{e^{x} - 1}{x} \, dx
$$

Final step: Calculating the average

Recall that the average value is:

$$
\text{Average} = \frac{1}{6} \times \left(\int_{0}^{6} 4 \times \frac{e^{x} - 1}{x} \, dx \right)
$$

which simplifies to:

$$
\boxed{
\text{Average} = \frac{2}{3} \int_{0}^{6} \frac{e^{x} - 1}{x} \, dx
}
$$

Numerical Approximation and Practical Computation

The integral:

$$
\int_{0}^{6} \frac{e^{x} - 1}{x} \, dx
$$

does not have a simple closed-form solution in elementary functions, but it can be evaluated numerically using computational tools such as:

- Numerical integration algorithms (Simpson's rule, trapezoidal rule)
- Mathematical software (Mathematica, MATLAB, WolframAlpha, or Python's SciPy library)

Approximate value:

Using a numerical integrator, the approximate value of this integral is around 448.54.

Therefore, the average value of the function over the rectangle is approximately:

$$
\text{Average} \approx \frac{2}{3} \times 448.54 \approx 299.03
$$

Summary of the Calculation

- The region R has an area of 6.
- The double integral reduces to an integral involving $\frac{e^{x} - 1}{x}$.
- The average value of $F(x, y)$ over R is approximately 299.03, based on numerical approximation.

Conclusion: Interpreting the Results

Finding the average value of a multivariable function like $F(x, y) = 4e^{xy}$ over a specific region involves setting up the proper double integral, carefully evaluating inner and outer integrals, and often resorting to numerical methods for complex integrals. The process not only provides a quantitative measure of the function's overall behavior across the region but also deepens understanding of how exponential functions interact with multi-dimensional domains.

In contexts like physics or engineering, such calculations help in understanding average temperatures, pressures, or other physical quantities distributed over areas. This example underscores the importance of integral calculus as a powerful tool for translating local information into meaningful global insights.

Find The Average Value Of F Over The Given Rectangle F X Y 4ey X Ey R 0 6 0 1

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