Find The Values Of At Which There Are Horizontal Tangent Lines On The Graph Of R = 1 + Cos .

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Introduction to the Problem

Understanding the behavior of polar curves is an essential aspect of calculus and analytical geometry. The function \(R = 1 + \cos \theta \) is a classic example of a polar equation that produces a well-known shape called a cardioid. One intriguing question in the analysis of such curves is identifying the points where the tangent line to the graph is horizontal. Specifically, we want to determine the values of \(\theta\) at which the tangent line to the curve \(R = 1 + \cos \theta \) is horizontal.

This problem involves calculating derivatives in polar coordinates and analyzing the conditions under which the slope of the tangent line becomes zero. Since the tangent line is horizontal when its slope is zero, our goal is to find the values of \(\theta\) that satisfy this condition.

Understanding the Polar Equation \(R = 1 + \cos \theta \)

Graphical Interpretation

The equation \(R = 1 + \cos \theta \) describes a well-known cardioid centered at the origin with its cusp at \(\theta = \pi\). As \(\theta\) varies from \(0\) to \(2\pi\), the radius \(R\) oscillates between:

- A minimum value of \(R_{\text{min}} = 1 + \cos \pi = 1 - 1 = 0 \),
- A maximum value of \(R_{\text{max}} = 1 + \cos 0 = 2 \).

The curve exhibits symmetry about the polar axis (\(\theta = 0\)) and is smooth everywhere except at the cusp point where \(R=0\).

Deriving the Conditions for Horizontal Tangents

Step 1: Convert Polar to Cartesian Coordinates

The first step involves understanding the slope of the tangent line in Cartesian coordinates, where the relationship between polar and Cartesian coordinates is:

$$
x = R \cos \theta = (1 + \cos \theta) \cos \theta,
$$
$$
y = R \sin \theta = (1 + \cos \theta) \sin \theta.
$$

However, directly differentiating these parametric equations can be cumbersome. It's more straightforward to use the polar derivative formulas.

Step 2: Find the Derivatives $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$

Given that

$$
x(\theta) = R(\theta) \cos \theta,
$$
$$
y(\theta) = R(\theta) \sin \theta,
$$

we can differentiate both with respect to θ:

$$
\frac{dx}{d\theta} = \frac{dR}{d\theta} \cos \theta - R \sin \theta,
$$
$$
\frac{dy}{d\theta} = \frac{dR}{d\theta} \sin \theta + R \cos \theta.
$$

Step 3: Calculate $\frac{dR}{d\theta}$

Since $R = 1 + \cos \theta$,

$$
\frac{dR}{d\theta} = - \sin \theta.
$$

Step 4: Express $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$

Substituting R and $\frac{dR}{d\theta}$:

$$
\frac{dx}{d\theta} = - \sin \theta \cos \theta - (1 + \cos \theta) \sin \theta,
$$

$$
\frac{dy}{d\theta} = - \sin \theta \sin \theta + (1 + \cos \theta) \cos \theta.
$$

Simplify each:

$$
\frac{dx}{d\theta} = - \sin \theta \cos \theta - \sin \theta - \cos \theta \sin \theta = - 2 \sin \theta \cos \theta - \sin \theta,
$$

$$
\frac{dy}{d\theta} = - \sin^2 \theta + (1 + \cos \theta) \cos \theta.
$$

Step 5: Find the Slope of the Tangent Line $\frac{dy}{dx}$

The slope of the tangent in Cartesian terms is:

$$
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.
$$

Substitute the simplified derivatives:

$$
\frac{dy}{dx} = \frac{- \sin^2 \theta + (1 + \cos \theta) \cos \theta}{- 2 \sin \theta \cos \theta - \sin \theta}.
$$

Finding When the Tangent is Horizontal

Condition for Horizontal Tangents

A tangent line is horizontal when:

\[

\frac{dy}{dx} = 0,
\]

which implies:

\[
\frac{dy}{d\theta} = 0,
\]

since the denominator \(\frac{dx}{d\theta} \neq 0 \) at those points (we will verify this).

Step 6: Solve \(\frac{dy}{d\theta} = 0 \)

Recall:

\[
\frac{dy}{d\theta} = - \sin^2 \theta + (1 + \cos \theta) \cos \theta.
\]

Expand and simplify:

\[
\frac{dy}{d\theta} = - \sin^2 \theta + \cos \theta + \cos^2 \theta.
\]

Express \(\sin^2 \theta \) in terms of \(\cos^2 \theta \):

\[
\sin^2 \theta = 1 - \cos^2 \theta,
\]

so

\[
\frac{dy}{d\theta} = - (1 - \cos^2 \theta) + \cos \theta + \cos^2 \theta = -1 + \cos^2 \theta + \cos \theta + \cos^2 \theta.
\]

Combine like terms:

\[
\frac{dy}{d\theta} = -1 + 2 \cos^2 \theta + \cos \theta.
\]

Set this equal to zero:

\[
-1 + 2 \cos^2 \theta + \cos \theta = 0.
\]

Rearranged:

\[
2 \cos^2 \theta + \cos \theta - 1 = 0.

\]

Let \(x = \cos \theta \). The quadratic becomes:

\[
2 x^2 + x - 1 = 0.
\]

Step 7: Solve the Quadratic Equation

Use the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\]

where \(a=2 \), \(b=1 \), \(c=-1 \).

Calculate discriminant:

\[
\Delta = 1^2 - 4 \times 2 \times (-1) = 1 + 8 = 9.
\]

Thus,

\[
x = \frac{-1 \pm \sqrt{9}}{2 \times 2} = \frac{-1 \pm 3}{4}.
\]

This yields:

- \(x = \frac{-1 + 3}{4} = \frac{2}{4} = \frac{1}{2} \),
- \(x = \frac{-1 - 3}{4} = \frac{-4}{4} = -1 \).

Recall \(x = \cos \theta \), so:

\[
\cos \theta = \frac{1}{2} \quad \text{or} \quad \cos \theta = -1.
\]

Determining the \(\theta\) Values for Horizontal Tangents

Case 1: \(\cos \theta = \frac{1}{2} \)

The solutions in \([0, 2\pi)\):

$$
\theta = \pm \frac{\pi}{3} + 2k\pi,
$$

which simplifies to:

$$
\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}.
$$

Case 2: $\cos \theta = -1$

The solution:

$$
\theta = \pi + 2k\pi,
$$

which in $[0, 2\pi)$:

$$
\boxed{\theta = \pi}.
$$

Verifying the Validity of These Points

Check the Derivative $\frac{dx}{d\theta}$ at the Solutions

Recall:

Frequently Asked Questions

How do I find the points where the graph of R = 1 + Cos θ has horizontal tangent lines?

To find where the graph has horizontal tangents, differentiate R with respect to θ and set the derivative of the y-component to zero, solving for θ to identify those points.

What is the derivative of R = 1 + Cos θ with respect to θ?

The derivative of R = 1 + Cos θ with respect to θ is dR/dθ = -Sin θ.

At which values of θ does the graph of R = 1 + Cos θ have horizontal tangents?

Horizontal tangents occur when the derivative of the y-component with respect to θ is zero, i.e., when Sin θ = 0, which happens at θ = 0, π, 2π, etc.

What are the specific θ-values for horizontal tangent lines within one period?

Within one period (0 to 2π), the graph has horizontal tangent lines at θ = 0, π, and 2π.

How do I interpret the θ-values where horizontal tangents occur in the graph of R = 1 + Cos θ?

These θ-values correspond to points on the polar graph where the radius reaches a maximum or minimum, and the graph has a horizontal tangent line at those points.

Can you summarize the process to find all θ-values with horizontal tangents for this polar function?

Yes. First, differentiate R with respect to θ, set the derivative of the y-component to zero (i.e., Sin θ = 0), solve for θ, and identify those points as where the graph has horizontal tangent lines.

Additional Resources

Understanding Horizontal Tangent Lines on the Graph of \(R = 1 + \cos \theta \)

When exploring the fascinating world of polar graphs, one of the key features that often captures the interest of mathematicians and students alike is the concept of tangent lines—particularly, horizontal tangents. The graph of \(R = 1 + \cos \theta \) is a classic example that offers rich insights into how curves behave in polar coordinates. In this article, we delve deep into the question: At which values of \(\theta \) does the graph of \(R = 1 + \cos \theta \) have horizontal tangent lines?

Our journey will be comprehensive, covering foundational concepts, mathematical derivations, and visual interpretations to ensure a thorough understanding of this intriguing problem.

Fundamentals of Polar Coordinates and Tangent Lines

Before tackling the specifics, it’s essential to revisit some core principles of polar coordinates and how tangent lines are characterized within this framework.

Polar Coordinates Overview

In polar coordinates, each point in the plane is described by two parameters:

- r: the distance from the origin to the point.
- θ: the angle measured from the positive x-axis to the line segment connecting the origin to the point.

A polar equation like $R = 1 + \cos \theta$ defines a locus of points where the radius r varies with the angle θ.

Converting Polar to Cartesian Coordinates

To analyze tangent lines, sometimes it’s helpful to convert polar equations into Cartesian coordinates:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

For the given function:

$$
r = 1 + \cos \theta
$$

we get:

$$
x = (1 + \cos \theta) \cos \theta, \quad y = (1 + \cos \theta) \sin \theta
$$

While this transformation aids visualization, for the purpose of finding tangent slopes, it’s often preferable to work directly with derivatives in polar form.

Mathematical Framework for Finding Horizontal Tangents

The core idea is to determine the points on the curve where the tangent line is horizontal. In polar coordinates, the slope of the tangent line at a point (θ, r) is given by the derivative $\frac{dy}{dx}$.

Deriving $\frac{dy}{dx}$ in Polar Coordinates

Given:

$$
x = r \cos \theta, \quad y = r \sin \theta

\]

The derivatives with respect to θ are:

$$
\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta
$$
$$
\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta
$$

Therefore, the slope $\frac{dy}{dx}$ is:

$$
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}
$$

Key Point: To find points where the tangent line is horizontal, set $\frac{dy}{dx} = 0$.

Step-by-Step Solution for $R = 1 + \cos \theta$

Let’s proceed with the specific function:

$$
r = 1 + \cos \theta
$$

Step 1: Compute $\frac{dr}{d\theta}$

$$
\frac{dr}{d\theta} = - \sin \theta
$$

Step 2: Write expressions for $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$

$$
\frac{dy}{d\theta} = (- \sin \theta) \sin \theta + (1 + \cos \theta) \cos \theta
$$
$$
= - \sin^2 \theta + (1 + \cos \theta) \cos \theta
$$

Similarly,

$$
\frac{dx}{d\theta} = (- \sin \theta) \cos \theta - (1 + \cos \theta) \sin \theta
$$

$$
= - \sin \theta \cos \theta - (1 + \cos \theta) \sin \theta
$$

Step 3: Simplify the derivatives

- For numerator:

$$
\frac{dy}{d\theta} = - \sin^2 \theta + \cos \theta + \cos^2 \theta
$$
$$
= (\cos^2 \theta - \sin^2 \theta) + \cos \theta
$$
$$
= \cos 2\theta + \cos \theta
$$

- For denominator:

$$
\frac{dx}{d\theta} = - \sin \theta \cos \theta - \sin \theta - \cos \theta \sin \theta
$$
$$
= - 2 \sin \theta \cos \theta - \sin \theta
$$
$$
= - \sin \theta (2 \cos \theta + 1)
$$

Step 4: Set the derivative $\frac{dy}{dx}$ to zero for horizontal tangents

$$
\frac{dy}{d\theta} = 0
$$

which implies:

$$
\cos 2\theta + \cos \theta = 0
$$

Solving for θ Corresponding to Horizontal Tangents

Now, focus on the equation:

$$
\cos 2\theta + \cos \theta = 0
$$

Recall the double-angle identity:

$$
\cos 2\theta = 2 \cos^2 \theta - 1
$$

Substitute into the equation:

$$
2 \cos^2 \theta - 1 + \cos \theta = 0
$$

Rearranged as:

$$
2 \cos^2 \theta + \cos \theta - 1 = 0
$$

Let $x = \cos \theta$:

$$
2x^2 + x - 1 = 0
$$

This is a quadratic in x:

$$
2x^2 + x - 1 = 0
$$

Apply quadratic formula:

$$
x = \frac{-1 \pm \sqrt{1^2 - 4 \times 2 \times (-1)}}{2 \times 2} = \frac{-1 \pm \sqrt{1 + 8}}{4} = \frac{-1 \pm \sqrt{9}}{4}
$$

$$
x = \frac{-1 \pm 3}{4}
$$

Thus,

- When taking the positive root:

$$
x = \frac{-1 + 3}{4} = \frac{2}{4} = \frac{1}{2}
$$

- When taking the negative root:

$$
x = \frac{-1 - 3}{4} = \frac{-4}{4} = -1
$$

Corresponding θ values:

- For $\cos \theta = \frac{1}{2}$:

$$
\theta = \pm \frac{\pi}{3} + 2\pi k, \quad k \in \mathbb{Z}

\]

- For \(\cos \theta = -1 \):

\[
\theta = \pi + 2\pi k, \quad k \in \mathbb{Z}
\]

Verifying the Denominator and Ensuring Validity

Recall that the derivative \(\frac{dy}{dx} = 0 \) when the numerator is zero, provided the denominator is not zero (which would indicate a vertical tangent or undefined slope).

The denominator:

\[
- \sin \theta (2 \cos \theta + 1)
\]

- At \(\theta = \pm \frac{\pi}{3} \):

\[
\sin \left(\pm \frac{\pi}{3} \right) = \pm \frac{\sqrt{3}}{2}
\]
\[
2 \cos \left(\pm \frac{\pi}{3} \right) + 1 = 2 \times \frac{1}{2} + 1 = 1 + 1 = 2
\]

Denominator becomes:

\[
- \left(\pm \frac{\sqrt{3}}{2} \right) \times 2 = - \pm \sqrt{3}
\]

which is non-zero. So, the tangent lines at these points are valid and horizontal.

- At \(\theta = \pi \), where \(\cos \pi = -1 \):

\[
\sin \pi = 0
\]

Thus, denominator:

\[
- 0 \times (2 \times -1 + 1) =

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