If A, B, And C Are Invertible Matrices, Does The Equation Have A Solution X? If So, Find It.

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When working with matrices in linear algebra, a common question is whether a certain matrix equation has a solution, and if it does, how to find it. Specifically, when matrices A, B, and C are all invertible, it often simplifies the process of solving equations involving these matrices. This article explores the conditions under which such equations have solutions, how the invertibility of matrices influences these solutions, and provides step-by-step methods to find the solution matrix X when possible.

Understanding Invertible Matrices and Their Properties

Before delving into the solution process, it’s essential to understand what makes a matrix invertible and what properties these matrices possess.

What Is an Invertible Matrix?

An invertible matrix, also known as a non-singular matrix, is a square matrix A for which there exists another matrix, denoted as A^{-1}, such that:

$$
A A^{-1} = A^{-1} A = I
$$

Here, I is the identity matrix of the same size as A. The inverse matrix essentially "undoes" the transformation represented by A.

Properties of Invertible Matrices

- Existence of Inverse: Only square matrices can be invertible.
- Determinant: A matrix is invertible if and only if its determinant is non-zero ($\det(A) \neq 0$).
- Multiplicative Property: The inverse of a product of invertible matrices satisfies $(AB)^{-1} = B^{-1}A^{-1}$.
- Solving Linear Systems: If A is invertible, the system $AX = B$ has a

unique solution $X = A^{-1}B$.

Formulating the Matrix Equation

Suppose we have an equation involving matrices A, B, C, and an unknown matrix X:

$$
A X B = C
$$

The goal is to determine whether this equation has a solution for X, given that A, B, and C are invertible matrices, and if so, to find X explicitly.

Does the Equation Have a Solution?

The invertibility of A, B, and C greatly influences the existence and uniqueness of the solution.

Impact of Invertibility on the Solution

- Since A and B are invertible, it suggests we can manipulate the equation to isolate X.
- The invertibility of C affects whether the equation is consistent or not, but generally, if A and B are invertible, a solution exists for any C.

Key Point: When A and B are invertible, the equation $A X B = C$ always has a unique solution for any C (including when C is invertible), because the invertibility allows "undoing" the transformations applied by A and B.

Finding the Solution X

Given the equation:

$$
A X B = C
$$

and knowing that A, B, C are invertible, we can solve for X as follows:

1. Multiply Both Sides by A^{-1} on the Left:

\[
A^{-1} (A X B) = A^{-1} C
\]

which simplifies to:

\[
(I) X B = A^{-1} C
\]

since \(A^{-1} A = I\).

2. Multiply Both Sides by \(B^{-1}\) on the Right:

\[
(I) X (B B^{-1}) = (A^{-1} C) B^{-1}
\]

which reduces to:

\[
X I = (A^{-1} C) B^{-1}
\]

and finally:

\[
X = A^{-1} C B^{-1}
\]

Result: The explicit solution for X is:

\[
\boxed{
X = A^{-1} C B^{-1}
}
\]

This formula indicates that once you have the inverses of A and B, and the matrix C, you can compute X directly.

Practical Example

Suppose the matrices are:

\[
A = \begin{bmatrix}
2 & 1 \\
0 & 1

\end{bmatrix},
\quad
B = \begin{bmatrix}
1 & 0 \\
3 & 2
\end{bmatrix},
\quad
C = \begin{bmatrix}
5 & 2 \\
4 & 3
\end{bmatrix}
\]

Step 1: Verify that A and B are invertible.

- Calculate \(\det(A) = (2)(1) - (0)(1) = 2 \neq 0\) → invertible.
- Calculate \(\det(B) = (1)(2) - (0)(3) = 2 \neq 0\) → invertible.

Step 2: Find \(A^{-1}\) and \(B^{-1}\).

- \(A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 1 \end{bmatrix}\)

- \(B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1.5 & 0.5 \end{bmatrix}\)

Step 3: Compute \(X = A^{-1} C B^{-1}\).

- First, compute \(A^{-1} C\):

\[
A^{-1} C = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix}
(0.5)(5) + (-0.5)(4) & (0.5)(2) + (-0.5)(3) \\
(0)(5) + (1)(4) & (0)(2) + (1)(3)
\end{bmatrix} = \begin{bmatrix}
2.5 - 2 & 1 - 1.5 \\
4 & 3
\end{bmatrix} = \begin{bmatrix}
0.5 & -0.5 \\
4 & 3
\end{bmatrix}
\]

- Then, multiply the result by \(B^{-1}\):

\[
X = \begin{bmatrix} 0.5 & -0.5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1.5 & 0.5 \end{bmatrix}

$$
\]

Calculating:

$$
X_{11} = (0.5)(1) + (-0.5)(-1.5) = 0.5 + 0.75 = 1.25
$$
$$
X_{12} = (0.5)(0) + (-0.5)(0.5) = 0 + (-0.25) = -0.25
$$
$$
X_{21} = (4)(1) + (3)(-1.5) = 4 - 4.5 = -0.5
$$
$$
X_{22} = (4)(0) + (3)(0.5) = 0 + 1.5 = 1.5
$$

So,

$$
\boxed{
X = \begin{bmatrix}
1.25 & -0.25 \\
-0.5 & 1.5
\end{bmatrix}
}
$$

This matrix X satisfies the original equation $A X B = C$.

Summary and Key Takeaways

- When matrices A, B, and C are invertible, the matrix equation $A X B = C$ always has a unique solution.
- The solution can be directly computed using the formula:

$$
X = A^{-1} C B^{-1}
$$

- The invertibility of A and B guarantees the existence of the inverses, simplifying the process of solving for X.
- Verifying invertibility via determinants is a crucial first step.
- The approach demonstrated in the example can be applied broadly to similar problems involving invertible matrices.

Additional Tips for Solving Matrix Equations

- Always verify invertibility before attempting to find the inverse.
- Use matrix multiplication carefully, respecting the order, as it's not commutative.
- When dealing with larger matrices, consider computational tools or software like MATLAB, NumPy (Python), or WolframAlpha for calculating inverses and

Frequently Asked Questions

If A, B, and C are invertible matrices, does the equation A X B = C always have a solution?

Yes, if A, B, and C are invertible matrices, then the equation A X B = C always has a unique solution.

What is the general solution for X in the equation A X B = C when A and B are invertible?

The solution is X = A^{-1} C B^{-1}.

Does the invertibility of A, B, and C guarantee a unique solution to the matrix equation?

Yes, the invertibility of A and B guarantees a unique solution for X, given by X = A^{-1} C B^{-1}.

If A and B are invertible, but C is not, can the equation A X B = C have a solution?

No, if C is not invertible, the equation may not have a solution, even if A and B are invertible.

How can we verify that the solution X = A^{-1} C B^{-1} is valid?

By substituting X back into the original equation, and confirming that A X B equals C.

Are there special cases where the solution to A X B = C is not unique despite invertibility?

No, when A and B are invertible, the solution is always unique and given by X = A^{-1} C B^{-1}.

If matrices A, B, and C are all invertible, is the solution to A X B = C always real-valued?

Yes, provided that A, B, and C are real invertible matrices, the solution X will also be a real matrix.

Can the equation A X B = C be used to solve for X in terms of A, B, and C if only A and B are invertible?

Yes, if A and B are invertible, then the solution is $X = A^{-1} C B^{-1}$ regardless of whether C is invertible.

Additional Resources

Invertibility of matrices and the solvability of matrix equations are fundamental concepts in linear algebra with profound implications across mathematics, engineering, computer science, and applied sciences. When matrices are invertible, they possess unique properties that significantly simplify the analysis of systems of linear equations. This article explores the question: If A, B, and C are invertible matrices, does the matrix equation $AX = B$ have a solution? If so, how can it be found? We will delve into the theoretical background, analyze various forms of matrix equations involving invertible matrices, and elucidate methods for solving such equations, highlighting the importance of invertibility in ensuring solutions exist and are unique.

Understanding Invertible Matrices

What Does It Mean for a Matrix to Be Invertible?

In linear algebra, a matrix M (of size $n \times n$) is said to be invertible or non-singular if there exists another matrix M^{-1} such that:

$$
MM^{-1} = M^{-1}M = I,
$$

where I is the identity matrix of the same size. The matrix M^{-1} is called the inverse of M.

Key properties of invertible matrices:

- Unique inverse: For a given invertible matrix M, the inverse M^{-1} is unique.
- Determinant condition: M is invertible if and only if $\det(M) \neq 0$.
- Full rank: M has rank n.

Implications of invertibility:

- The linear transformation associated with M is bijective.
- Systems of linear equations involving M have unique solutions.

Matrix Equations and Invertibility

General Forms of Matrix Equations

Matrix equations are expressions involving matrices and unknown matrices that need to be solved for the unknowns. Common forms include:

- Linear matrix equations: $AX = B$
- Sylvester equations: $AX + XB = C$
- Other polynomial or functional matrix equations

In the context of this discussion, the primary concern is the equation:

$$
AX = B,
$$

where A, B, and C are given matrices, and X is the matrix to be determined.

Impact of Invertibility on Solvability

When matrices involved are invertible, the solution process becomes straightforward:

- If A is invertible, then the equation $AX = B$ has a unique solution:

$$
X = A^{-1} B.

\]

- Similarly, if the equation involves other invertible matrices, their properties can be exploited to manipulate and solve the equations efficiently.

Key insight: The invertibility of matrices like A, B, and C directly influences whether the matrix equation has solutions and whether those solutions are unique.

Analyzing the Equation $AX = B$

Case 1: A is Invertible

Suppose matrices A and B are given, with A invertible. Then:

$$
AX = B
$$

can be solved as:

$$
X = A^{-1} B.
$$

This solution is guaranteed to exist and be unique because invertibility ensures that left-multiplication by A^{-1} is well-defined and bijective.

Implications:

- Existence: Since A is invertible, for any B, the product $A^{-1} B$ exists.
- Uniqueness: The solution $X = A^{-1} B$ is the only solution.

Example:

Let

$$
A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.
$$

Calculate A^{-1}:

$$
\det(A) = 2 \times 4 - 1 \times 3 = 8 - 3 = 5 \neq 0,
$$

so A is invertible. The inverse:

$$
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}.
$$

Thus,

$$
X = A^{-1} B = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 11 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 \times 5 + (-1) \times 11 \\ -3 \times 5 + 2 \times 11 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 20 - 11 \\ -15 + 22 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 1.4 \end{bmatrix}.
$$

Case 2: A is Not Invertible

If A is singular (not invertible), then the solution to $AX = B$ depends on the compatibility of B with the column space of A:

- Existence: A solution exists if and only if B lies in the column space of A.
- Uniqueness: If a solution exists, it is not necessarily unique; there could be infinitely many solutions or none.

Conditions for existence:

$$
B \in \operatorname{Col}(A),
$$

which means that B can be expressed as a linear combination of the columns of A.

Testing for solutions:

- Use methods like Gaussian elimination to determine if the system is consistent.

- If consistent, solutions can be found via the pseudo-inverse or parametric forms.

Extending to Equations Involving Multiple Invertible Matrices

Equation: $A X C = B$

When the matrix equation involves multiple matrices, such as:

$$
A X C = B,
$$

and A, C are invertible, the solution process involves manipulating the equation to isolate X.

Method:

$$
A X C = B
$$

Multiply both sides from the left by A^{-1}:

$$
A^{-1} A X C = A^{-1} B \Rightarrow X C = A^{-1} B.
$$

Next, multiply both sides from the right by C^{-1}:

$$
X C C^{-1} = A^{-1} B C^{-1} \Rightarrow X = A^{-1} B C^{-1}.
$$

Result:

$$
\boxed{
X = A^{-1} B C^{-1}
}
$$

Implications:

- The invertibility of A and C guarantees a unique solution.
- The solution explicitly involves the inverses of A and C.

Example:

Suppose

$$
A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.
$$

Compute A^{-1}:

$$
A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},
$$

and C^{-1}:

$$
C^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.
$$

Then,

$$
X = A^{-1} B C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.
$$

Proceed with the multiplications step-by-step to find the explicit X.

Implications and Applications in Practice

Solving Systems of Linear Equations

The invertibility of matrices directly

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