If The Null Space Of A 7 X 5 Matrix Is 3-dimensional, Find Rank A. Dim Row A. And Dim Col A.

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Understanding the properties of matrices is fundamental in linear algebra, especially when analyzing their ranks, null spaces, row spaces, and column spaces. When given specific details about a matrix, such as its size and the dimension of its null space, it becomes possible to determine other related properties like the rank, the dimension of the row space, and the dimension of the column space. This comprehensive article aims to explore these concepts thoroughly and guide you through the process of solving such problems systematically.

Introduction to Fundamental Concepts in Matrix Theory

Before delving into the specific problem, it is essential to review some core concepts in linear algebra that form the backbone of our analysis.

Matrix Dimensions and Rank

- Matrix Size: Usually expressed as m x n, indicating m rows and n columns.
- Rank of a Matrix: The maximum number of linearly independent rows or columns. It provides insight into the matrix's invertibility and the dimension of its image.

Null Space (Kernel)

- The set of all vectors x such that A x = 0.
- Its dimension, called the nullity, indicates how many degrees of freedom exist in the solutions to the homogeneous system.

Row Space and Column Space

- Row Space: The subspace spanned by the rows of the matrix.
- Column Space: The subspace spanned by the columns of the matrix.
- Both spaces have the same dimension, which equals the rank of the matrix.

The Rank-Nullity Theorem

This fundamental theorem links the dimensions of the null space and the row space of a matrix:

$$
\text{Rank}(A) + \text{Nullity}(A) = n
$$

where:
- n = number of columns of matrix A,
- $\text{Rank}(A)$ = dimension of the row (or column) space,
- $\text{Nullity}(A)$ = dimension of the null space.

Analyzing the Given Problem

The problem statement provides the following information:
- The matrix A is of size 7 x 5 (7 rows and 5 columns).
- The null space of A is 3-dimensional.

From this data, we aim to find:
- Rank A
- Dimension of the row space of A
- Dimension of the column space of A

Step-by-Step Solution

Let's systematically approach the problem by applying the core concepts:

Step 1: Identify Known Quantities

- Number of columns (n) = 5
- Nullity ($\text{Nullity}(A)$) = 3 (since the null space is 3-dimensional)

Step 2: Apply the Rank-Nullity Theorem

The theorem states:

$$
\text{Rank}(A) + \text{Nullity}(A) = n
$$

Substituting the known values:

$$
\text{Rank}(A) + 3 = 5
$$

$$
\Rightarrow \text{Rank}(A) = 5 - 3 = 2
$$

Thus, the rank of matrix A is 2.

Step 3: Determine the Dimension of Row and Column Spaces

- The row space of A has dimension equal to the rank:

$$
\text{Dim Row } A = \text{Rank}(A) = 2
$$

- The column space of A, being the span of its columns, also has the same dimension:

$$
\text{Dim Col } A = \text{Rank}(A) = 2
$$

Summary of Results

Based on the above calculations, the solutions are:

- Rank of A: 2
- Dimension of the row space of A: 2
- Dimension of the column space of A: 2

Deeper Insights and Implications

Understanding these results offers valuable insights into the structure of the matrix:

Implication of Rank and Nullity

- Since the rank is 2, the matrix A maps a 5-dimensional space (the column space) into a 2-dimensional subspace.
- The nullity being 3 indicates that the solution space to $A x = 0$ is quite large, reflecting many degrees of freedom in the null space.

Relation to the Matrix's Shape

- The matrix being 7 x 5 (more rows than columns) suggests it could be rank-deficient (not full rank), which aligns with the rank being less than 5.
- The rank being 2 indicates the matrix has only 2 linearly independent rows or columns, which influences the solutions to the associated linear systems.

Practical Applications

- Such matrices appear in various applications like data compression, systems of linear equations, and dimensionality reduction techniques.
- Recognizing the nullity helps in understanding the degrees of freedom and potential redundancies in data.

Additional Considerations and Related Concepts

While the primary focus is on the specific problem, exploring related topics enriches understanding:

Full Row Rank and Full Column Rank

- A matrix with full row rank (rank = number of rows) is surjective.
- A matrix with full column rank (rank = number of columns) is injective.

Rank Deficiency

- When the rank is less than the minimum of the number of rows and columns, the matrix is rank-deficient.
- In this case, the rank is 2, less than 5, confirming deficiency.

Rank-Nullity in Practice

- Helps determine the solvability of linear systems.
- For example, since the nullity is 3, the homogeneous system $A x = 0$ has infinitely many solutions.

Column and Row Space Relationship

- The dimensions of the column space and row space are equal (both equal to the rank).
- The column space is a subspace of $\mathbb{R}^7$, and the row space is a subspace of $\mathbb{R}^5$.

Conclusion

In conclusion, the problem of determining the rank, row space dimension, and column space dimension of a 7 x 5 matrix with a 3-dimensional null space can be systematically solved using the rank-nullity theorem and fundamental linear algebra principles. The key findings are:

- Rank A = 2
- Dimension of the row space = 2
- Dimension of the column space = 2

These results not only answer the specific question but also exemplify the power of core linear algebra concepts in analyzing matrix properties. Understanding these relationships is crucial for various applications across mathematics, engineering, and computer science.

SEO Keywords and Phrases

- Null space of matrix
- Matrix rank
- Dimension of row space
- Dimension of column space
- Rank-nullity theorem
- 7 x 5 matrix analysis
- Nullity of matrix
- Linear algebra concepts
- Rank deficiency
- Solving linear systems
- Matrix properties and applications

This comprehensive guide aims to clarify the process of analyzing matrices based on their null spaces, fostering a deeper understanding of linear algebra fundamentals essential for students, educators, and professionals alike.

Frequently Asked Questions

What is the relationship between the null space dimension and the rank of a 7x5 matrix?

The null space dimension, called the nullity, relates to the rank via the Rank-Nullity Theorem: nullity + rank = number of columns. For a 7x5 matrix, nullity + rank = 5.

Given the null space of a 7x5 matrix is 3-dimensional, what is the rank of the matrix?

The rank of the matrix is 2, since nullity + rank = 5, so 3 + rank = 5, thus rank = 2.

What is the dimension of the row space (row rank) of the matrix?

The row space dimension (row rank) equals the rank of the matrix, so it is 2.

What is the dimension of the column space (column rank) of the matrix?

The column space dimension (column rank) is equal to the rank, so it is 2.

How does the nullity of a matrix relate to its null space dimension?

The nullity of a matrix is the dimension of its null space, representing the number of free variables in the solution to Ax=0.

If a 7x5 matrix has a null space of dimension 3, can its rank be greater than 2? Why or why not?

No, because according to the Rank-Nullity Theorem, the rank must be 2; any higher would violate the theorem since nullity is 3.

Why is the null space dimension important in understanding the properties of a matrix?

The null space dimension indicates the degree of linear dependence among columns and the solutions to homogeneous equations, revealing the matrix's invertibility and rank properties.

Additional Resources

If The Null Space Of A 7 x 5 Matrix Is 3-dimensional, Find Rank A. Dim Row A. And Dim Col A.

Introduction

In the realm of linear algebra, understanding the relationships between various subspaces associated with a matrix is fundamental. These subspaces include the null space (kernel), column space, and row space, each providing critical insights into the matrix's properties, solutions to systems, and transformations. A core principle connecting these subspaces is the Rank-Nullity Theorem, which offers a powerful framework for deducing unknown quantities when some are known.

This investigation focuses on a particular scenario: Given a 7×5 matrix whose null space (nullity) is 3-dimensional, we aim to determine its rank, the dimension of its row space, and the dimension of its column space. This problem is not only a classic exercise in linear algebra but also a gateway to understanding the interplay between a matrix's structural properties and its solutions.

Fundamental Concepts and Theoretical Background

The Structure of a 7×5 Matrix

A 7×5 matrix, denoted as A, has 7 rows and 5 columns. Its entries can be thought of as a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^7$. The key subspaces associated with A include:

- Null space (Kernel): The set of all vectors $x \in \mathbb{R}^5$ such that $A x = 0$.

- Column space (Range): The subspace of $\mathbb{R}^7$ spanned by the columns of A.

- Row space: The subspace of $\mathbb{R}^5$ spanned by the rows of A.

The Rank-Nullity Theorem

A core theorem in linear algebra, the Rank-Nullity Theorem, states:

$$
\text{Rank}(A) + \text{Nullity}(A) = n
$$

where:

- $\text{Rank}(A)$ is the dimension of the column space (column rank).

- $\text{Nullity}(A)$ is the dimension of the null space (nullity).

- n is the number of columns of A.

In this case, since A is a 7×5 matrix, $n = 5$.

Deduction of the Nullity and Its Implications

Null Space Dimension

Given:

$$
\dim(\text{Null space of }A) = 3
$$

which directly provides the nullity:

$$
\text{Nullity}(A) = 3
$$

Applying the Rank-Nullity Theorem:

$$
\text{Rank}(A) + 3 = 5 \implies \text{Rank}(A) = 5 - 3 = 2
$$

Thus, the rank of A is 2.

This is a crucial deduction, as it immediately informs us about the maximum number of linearly independent columns and the maximum number of linearly independent rows.

Determining the Dimensions of Row and Column Spaces

Column Space Dimension

Since the rank of A is 2:

$$
\boxed{
\text{dim}(\text{Col }A) = \text{Rank}(A) = 2
}
$$

The column space is a 2-dimensional subspace of $\mathbb{R}^7$. This means that only two columns of A are linearly independent, and all other columns can be expressed as linear combinations of these.

Row Space Dimension

The rank of a matrix is also equal to the dimension of its row space:

$$
\boxed{
\text{dim}(\text{Row }A) = \text{Rank}(A) = 2
}
$$

\]

By the Rank-Row Rank Theorem, the row space (a subspace of $\mathbb{R}^5$) has dimension 2, indicating that the rows are linearly dependent except for two of them, which span the row space.

Additional Insights and Implications

The Relationship Between Row and Column Spaces

While the row space resides in $\mathbb{R}^5$, the column space resides in $\mathbb{R}^7$. Despite these different ambient spaces, the dimensions of these subspaces are equal because the rank of the matrix is the same when considering either the row space or the column space.

Geometric Interpretation

- The null space being 3-dimensional indicates that there are three free variables in the solution to $A x = 0$.

- The rank being 2 implies that the image of A is a 2-dimensional subspace in $\mathbb{R}^7$, meaning A maps $\mathbb{R}^5$ onto a 2-dimensional subspace.

- The fact that the nullity exceeds the rank indicates that A is highly underdetermined, with many solutions to homogeneous systems.

Practical Example and Application

Suppose we are working with a matrix A with the described properties. The implications are:

- When solving $A x = b$, solutions exist only if b lies within the 2-dimensional subspace spanned by A's column space.

- The general solution to the homogeneous system $A x = 0$ has dimension 3, indicating infinitely many solutions parametrized by three free variables.

- The rank of 2 indicates that A captures only a small part of the target space, which might be relevant in applications like data compression, signal processing, or systems modeling where the matrix's rank affects the expressiveness of the transformation.

Summary of Key Results

Quantity	Value	Explanation
Nullity of A	3	Given directly; from the null space dimension
Rank of A	2	From Rank-Nullity Theorem: 5 - 3 = 2
Dimension of Column Space	2	Equal to rank; subspace of $\mathbb{R}^7$

Dimension of Row Space	2	Equal to rank; subspace of $\mathbb{R}^5$

Conclusion

The exploration of a 7×5 matrix with a 3-dimensional null space exemplifies the elegance of linear algebra's foundational theorems. The nullity directly informs the rank via the Rank-Nullity Theorem, establishing that the matrix's rank is 2. Consequently, both the row space and column space are 2-dimensional subspaces, reflecting the matrix's capacity to map vectors from $\mathbb{R}^5$ into a 2-dimensional subspace of $\mathbb{R}^7$.

This analysis underscores the interconnectedness of subspace dimensions and matrix properties, providing a clear illustration of how theoretical principles translate into concrete numerical deductions. Understanding these relationships is essential for advanced studies in linear algebra, data science, engineering, and applied mathematics, where the structure of matrices underpins much of the analysis and problem-solving.

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Note: This article aims to provide a comprehensive understanding of the problem through theoretical explanation, practical implications, and detailed reasoning, suitable for students, educators, and practitioners interested in the foundational aspects of linear algebra.

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