Let S Be The Triangle With Vertices (0,1), (-1,0) And (1,0) In R. Find The Polar S Of S.

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Understanding the concept of polarity in geometry, especially in the context of convex sets such as triangles, is fundamental for advanced studies in convex analysis, optimization, and computational geometry. In this article, we will explore the process of finding the polar set of a specific triangle S in \(\mathbb{R}^2\), with vertices at (0,1), (-1,0), and (1,0). We will delve into the definitions, properties, and step-by-step procedures necessary for deriving the polar set \(S^\), providing comprehensive insights suitable for students and professionals alike.

Introduction to Polar Sets in \(\mathbb{R}^2\)

Before tackling the specific problem, it is essential to understand what polar sets are, their significance, and how they relate to convex sets.

Definition of the Polar Set

Given a set \(S \subseteq \mathbb{R}^n\), the polar set \(S^\) is defined as:

\[
S^ = \{ y \in \mathbb{R}^n \mid \sup_{x \in S} \langle x, y \rangle \leq 1 \}
\]

where \(\langle x, y \rangle\) denotes the dot product of vectors \(x\) and \(y\).

Key points:

- The polar set is always a convex, closed, and star-shaped set.
- If \(S\) is convex, bounded, and contains the origin, then \(S^\) is also convex.
- The operation of taking the polar set is involutive under certain conditions, meaning \((S^)^ = \overline{\text{conv}}(S)\), the closed convex hull of \(S\).

Importance of Polar Sets in Geometry and Optimization

Polar sets are instrumental in:

- Duality theory in convex optimization.
- Analyzing support functions and gauge functions.
- Studying dual norms and dual cones.

- Solving problems involving convex hulls and support hyperplanes.

Problem Setup: The Triangle \(S\)

Given the vertices:

- \(A = (0, 1)\)
- \(B = (-1, 0)\)
- \(C = (1, 0)\)

we define the set:

\[
S = \text{Convex hull of } \{A, B, C\}
\]

which forms a triangle in \(\mathbb{R}^2\).

Geometric Description of \(S\)

The triangle \(S\) can be described as the set of all convex combinations:

\[
S = \left\{ x = \lambda_A A + \lambda_B B + \lambda_C C \;\middle|\; \lambda_A, \lambda_B, \lambda_C \geq 0,\; \lambda_A + \lambda_B + \lambda_C = 1 \right\}
\]

Alternatively, \(S\) can be characterized by the intersection of half-planes defined by its edges.

Step-by-Step Approach to Find the Polar \(S^\)

The process involves multiple steps:

1. Identify the boundary lines (edges) of \(S\).
2. Express \(S\) via inequalities (half-plane representations).
3. Use the inequalities to derive the polar set \(S^\).

Let's proceed systematically.

Step 1: Find the Equations of the Edges of S

The triangle has three edges:

- Edge AB
- Edge BC
- Edge CA

Edge AB: between $A = (0, 1)$ and $B = (-1, 0)$

- Vector $AB = B - A = (-1, 0) - (0, 1) = (-1, -1)$

Equation of line AB:

$$
\text{Parameterize: } x = 0 + t(-1), \quad y = 1 + t(-1)
$$

or directly find the line through A and B:

$$
\text{Slope } m_{AB} = \frac{0 - 1}{-1 - 0} = \frac{-1}{-1} = 1
$$

Line equation:

$$
y - 1 = 1(x - 0) \implies y = x + 1
$$

Edge BC: between $B = (-1, 0)$ and $C = (1, 0)$

- It's a horizontal line at $y=0$:

$$
y = 0
$$

Edge CA: between $C = (1, 0)$ and $A = (0, 1)$

- Slope:

$$
m_{CA} = \frac{1 - 0}{0 - 1} = \frac{1}{-1} = -1
$$

Line equation:

$$
y - 0 = -1(x - 1) \implies y = -x + 1
$$

$$

Step 2: Express S via Inequalities (Half-Planes)

Since S is the convex hull of the three vertices, it can be described as the intersection of the half-planes that contain the interior of the triangle.

For each edge:

- Determine the inequality that holds for all points inside S.

Edge AB: line $y = x + 1$

- Inside S, the points are on the side of the line that contains the vertex $C = (1, 0)$.

- Plugging $C = (1, 0)$ into $y - x - 1$:

$$
0 - 1 - 1 = -2 < 0
$$

Thus, the interior of S is on the side where:

$$
y - x - 1 \leq 0
$$

or equivalently:

$$
y \leq x + 1
$$

Edge BC: line $y=0$

- Inside S, contains vertices $A=(0,1)$:

$$
1 > 0
$$

- Since A is above the line $y=0$, the interior lies where:

$$
y \geq 0

\]

Edge \(CA\): line \(y = -x + 1\)

- Check point \(B = (-1, 0)\):

\[
0 - (-1) + 1 = 0 + 1 + 1 = 2 > 0
\]

- The interior is on the side where:

\[
y \leq -x + 1
\]

Summary of inequalities:

\[
\begin{cases}
y \leq x + 1 \quad &\text{(from edge \(AB\))} \\
y \geq 0 \quad &\text{(from edge \(BC\))} \\
y \leq -x + 1 \quad &\text{(from edge \(CA\))}
\end{cases}
\]

Thus,

\[
S = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y \geq 0,\ y \leq x + 1,\ y \leq -x + 1 \right\}
\]

Step 3: Derive the Polar Set \(S^\)

Recall the definition:

\[

$$
S^ = \{ y \in \mathbb{R}^2 \mid \sup_{x \in S} \langle x, y \rangle \leq 1 \}
$$

Since S is the intersection of half-planes, the support function $h_S(y)$ is:

$$
h_S(y) = \sup_{x \in S} \langle x, y \rangle
$$

and the polar set is:

$$
S^ = \{ y \in \mathbb{R}^2 \mid h_S(y) \leq 1 \}
$$

Therefore, to find $S^$, we need to compute the support function $h_S(y)$ for arbitrary $y = (p, q)$, and impose the condition $h_S(y) \leq 1$.

Calculating the Support Function $h_S(y)$

Since S is the intersection of the three half-planes, its support function is the minimum of the support functions of these half-planes.

But more straightforwardly, since S is a convex polygon with vertices A, B, C, the support function can be computed as:

$$
h_S(y) = \max_{x \in \text{vertices}} \langle x, y \rangle
$$

which simplifies to:

\[
h_S(y

Frequently Asked Questions

What is the definition of the polar set S of a given triangle in R?

The polar set S of a triangle in R consists of all points y such that the inner product with every point in the triangle is less than or equal to 1; mathematically, S = { y ∈ R | ⟨x, y⟩ ≤ 1 for all x ∈ S }.

Given the vertices of triangle S as (0,1), (-1,0), and (1,0), how do we determine its polar set S?

To find the polar S, we set up inequalities for each vertex: for $y = (a, b)$, ensure $\langle(0,1), (a,b)\rangle \leq 1$, $\langle(-1,0), (a,b)\rangle \leq 1$, and $\langle(1,0), (a,b)\rangle \leq 1$, then find the intersection of the resulting half-planes.

What are the inequalities defining the polar set S for the given triangle vertices?

The inequalities are: $b \leq 1$ (from $(0,1)$), $-a \leq 1$ (from $(-1,0)$), and $a \leq 1$ (from $(1,0)$).

What is the geometrical shape of the polar set S for the given triangle?

The polar set S is a triangle in R defined by the intersection of the half-planes: $a \leq 1$, $a \geq -1$, and $b \leq 1$, which forms a rectangle or a bounded polygon depending on the constraints.

How can we explicitly describe the vertices of the polar triangle S?

The vertices of S are at the intersections of the boundary lines: $(-1, 1)$, $(1, 1)$, and the points where a and b meet these constraints, forming a triangle with vertices at $(-1, 1)$, $(1, 1)$, and possibly at $(0, -\infty)$ if unbounded, but in this case, the bounded region is between $a = -1$ and $a = 1$, and $b \leq 1$.

Why is understanding the polar of a triangle important in convex analysis?

The polar provides insight into duality, optimization problems, and the geometric relationships between convex sets, enabling easier analysis of constraints and feasible regions in various applications.

Additional Resources

Let S Be The Triangle With Vertices (0,1), (-1,0), And (1,0) In R. Find The Polar S Of S.

Introduction

Let S be the triangle with vertices (0,1), (-1,0), and (1,0) in R. Find the polar S of S. This problem is situated within the fascinating domain of convex geometry, a branch of mathematics that explores the properties of convex sets and their duals. The concept of polar sets (or polars) is fundamental in this area, providing a dual perspective that often simplifies complex geometric problems and reveals deep structural insights.

In this article, we will undertake a comprehensive exploration of the problem, starting from the foundational definitions of convex sets and polars, moving through the specific properties of the given triangle, and culminating in a detailed calculation of its polar set. Our approach aims to be accessible to readers with a basic understanding of geometry and linear algebra, yet sufficiently rigorous to elucidate the nuances involved in the process.

Section 1: Fundamental Concepts

1.1 Convex Sets and Convex Hulls

A set $S \subseteq \mathbb{R}^n$ is convex if, for any two points $x, y \in S$, the line segment connecting them is entirely contained within S. Formally,

$$
\forall x, y \in S, \quad \lambda x + (1 - \lambda) y \in S, \quad \text{for all } \lambda \in [0,1].
$$

The convex hull of a set of points is the smallest convex set containing all those points. For a finite set of points, the convex hull is essentially the "minimal" convex polygon or polyhedron enclosing the points.

1.2 Definition of the Polar Set

Given a convex set \(S \subseteq \mathbb{R}^n \) that contains the origin (or is contained within some bounded region), its polar set \(S^ \) is defined as:

\[
S^ = \{ y \in \mathbb{R}^n : \sup_{x \in S} \langle y, x \rangle \leq 1 \}.
\]

Here, \(\langle y, x \rangle \) denotes the standard inner product in \(\mathbb{R}^n \). Intuitively, the polar set comprises all vectors \(y \) such that the inner product with any point in \(S \) does not exceed 1.

The polar set is always convex, closed, and contains the origin if \(S \) is convex and contains the origin. If \(S \) is a convex body containing the origin in its interior, then \((S^)^ = S \).

Section 2: The Geometry of the Triangle S

2.1 Coordinates and Basic Properties

The triangle \(S \) is specified by vertices:

- $A = (0, 1)$,
- $B = (-1, 0)$,
- $C = (1, 0)$.

Plotting these points, the triangle is symmetric with respect to the y-axis and lies above the x-axis, with the base along the segment from $(-1, 0)$ to $(1, 0)$ and the apex at $(0, 1)$.

Key properties:

- Vertices: A, B, C.
- Edges:
- AB: from $(0,1)$ to $(-1,0)$.
- AC: from $(0,1)$ to $(1,0)$.
- BC: from $(-1,0)$ to $(1,0)$.

- Convexity: Since S is a triangle, it is convex.

- Containment of the origin: The origin $(0,0)$ lies inside S. To verify:

The point $(0,0)$ is inside if it can be expressed as a convex combination of A, B, C:

$$
(0,0) = \alpha A + \beta B + \gamma C,
$$
with $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$.

Solving:

$$
(0,0) = \alpha (0,1) + \beta (-1,0) + \gamma (1,0),
$$

$$
\Rightarrow 0 = \beta (-1) + \gamma (1),
$$
$$
0 = \alpha (1),
$$
$$
\Rightarrow \alpha = 0,
$$
$$
\Rightarrow -\beta + \gamma = 0 \Rightarrow \gamma = \beta,
$$
$$
\text{and } \alpha + \beta + \gamma = 1 \Rightarrow 0 + \beta + \beta = 1 \Rightarrow 2 \beta = 1 \Rightarrow \beta = \frac{1}{2},
$$

$$
\Rightarrow \gamma = \frac{1}{2}.
$$

Since $\beta, \gamma \geq 0$, and $\alpha=0$, the

origin is on the boundary (specifically, the midpoint of the base \(BC \)). Thus, the origin lies within or on the boundary of \(S \).

Section 3: Computing the Polar S of S

3.1 Approach and Strategy

To find the polar set \(S^ \), we leverage the definition:

\[
S^ = \{ y \in \mathbb{R}^2 : \sup_{x \in S} \langle y, x \rangle \leq 1 \}.
\]

Since \(S \) is a polygon (triangle), the supremum over \(x \in S \) of \(\langle y, x \rangle \) is achieved at one of the vertices or along an edge, because linear functions attain maxima on convex polygons at vertices or edges.

Method:

1. Identify the support function \(h_S(y) \):

\[
h_S(y) = \sup_{x \in S} \langle y, x \rangle,

\]

which equals the maximum of \(\langle y, x \rangle \) over the vertices and edges.

2. Determine the inequalities that define \(S^ \):

\[
\text{For each } y \in S^, \quad h_S(y) \leq 1,
\]

which is equivalent to:

\[
\langle y, x_i \rangle \leq 1, \quad \text{for all vertices } x_i \in \{A, B, C\}.
\]

3. Express these inequalities explicitly and find the intersection:

The set \(S^ \) is the intersection of half-planes:

\[
\{ y : \langle y, A \rangle \leq 1 \}, \quad \{ y : \langle y, B \rangle \leq 1 \}, \quad \{ y : \langle y, C \rangle \leq 1 \}.
\]

This process reduces the problem to solving a system of linear inequalities to describe $S^$.

3.2 Calculating the Support Function

Let's compute $h_S(y)$ explicitly:

$$
h_S(y) = \max_{x \in S} \langle y, x \rangle.
$$

Since S is convex and polygonal, the maximum is achieved at vertices:

$$
h_S(y) = \max \{\langle y, A \rangle, \langle y, B \rangle, \langle y, C \rangle \}.
$$

Expressed explicitly:

- $\langle y, A \rangle = y_1 \times 0 + y_2 \times 1 = y_2$,
- $\langle y, B \rangle = y_1 \times (-1) + y_2 \times 0 = - y_1$,
- $\langle y, C \rangle = y_1 \times 1 + y_2 \times 0 = y_1$.

Thus,

\[

h_S(y) = \max \{ y_2, - y_1, y_1 \}.
\]

Section 4: Deriving the Polar Set \(S^ \)

Given the support function:

\[
h_S(y) = \max \{ y_2, - y_1, y_1 \},
\]

the polar set \(S^ \) is:

\[
S^ = \{ y \in \mathbb{R}^2 : h_S(y) \leq 1 \}.
\]

[Let S Be The Triangle With Vertices 0 1 1 0 And 1 0 In R Find The Polar S Of S]

Let S Be The Triangle With Vertices 0 1 1 0 And 1 0 In R Find The Polar S Of S

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