Number Theory5. Find All Integer Solutions X, Y Such That 3? 7y2 = 1. Justify Your Answer! -

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Introduction to the Problem

Number theory is a fascinating branch of mathematics dealing with the properties and relationships of integers. A common type of problem involves finding integer solutions to algebraic equations. In this article, we explore the problem:

Find all integer solutions (X, Y) such that 3? $7Y^2 = 1$. Justify your answer!

While the notation "3?" may seem ambiguous, it often suggests an operation or a placeholder for an unknown function or operator. For clarity, we interpret the problem as:

Find all integers X and Y such that $3X + 7Y^2 = 1$.

This is a linear Diophantine equation involving two variables. Our goal is to determine all integer solutions and justify the methods used to find these solutions.

Understanding the Equation

The equation:

\[3X + 7Y^2 = 1 \]

is a linear Diophantine equation in two variables, X and Y. To find solutions, we need to analyze the conditions under which X and Y are integers satisfying this relation.

Key points:

- The term \(7Y^2\) is always non-negative since squares are non-negative.
- The term \(3X\) can be positive, negative, or zero.
- Our task is to find integer pairs \((X, Y)\) satisfying the equation.

Rearranging the Equation

To analyze solutions systematically, rewrite the equation as:

$$ 3X = 1 - 7Y^2 $$

or equivalently,

$$ X = \frac{1 - 7Y^2}{3} $$

For X to be an integer, the numerator $1 - 7Y^2$ must be divisible by 3.

Divisibility Conditions and Modular Arithmetic

The key to finding integer solutions is to analyze the divisibility condition:

$$ 3 \mid (1 - 7Y^2) $$

which means:

$$ 1 - 7Y^2 \equiv 0 \pmod{3} $$

or:

$$ 7Y^2 \equiv 1 \pmod{3} $$

Since $7 \equiv 1 \pmod{3}$, the congruence simplifies to:

$$ 1 \cdot Y^2 \equiv 1 \pmod{3} $$

or:

$$ Y^2 \equiv 1 \pmod{3} $$

Possible Values of Y modulo 3

The quadratic residues modulo 3 are:

- $Y \equiv 0 \pmod{3} \Rightarrow Y^2 \equiv 0 \pmod{3}$
- $Y \equiv 1 \pmod{3} \Rightarrow Y^2 \equiv 1 \pmod{3}$
- $Y \equiv 2 \pmod{3} \Rightarrow Y^2 \equiv 4 \equiv 1 \pmod{3}$

Therefore, the solutions for Y modulo 3 are:

- $Y \equiv 1 \pmod{3}$
- $Y \equiv 2 \pmod{3}$

In both cases, $Y^2 \equiv 1 \pmod{3}$, satisfying the divisibility condition.

Conclusion: For all integers Y such that $Y \equiv 1 \pmod{3}$ or $Y \equiv 2 \pmod{3}$, the numerator $1 - 7Y^2$ is divisible by 3, and thus X is an integer.

Explicit Solutions for X and Y

Given that Y must satisfy $Y \equiv 1 \pmod{3}$ or $Y \equiv 2 \pmod{3}$, we can express:

$$ Y = 3k + r $$

where:

- $k \in \mathbb{Z}$,
- $r = 1$ or $r = 2$.

Now, compute X:

$$ X = \frac{1 - 7Y^2}{3} $$

Substitute $Y = 3k + r$:

$$ Y^2 = (3k + r)^2 = 9k^2 + 6kr + r^2 $$

Thus:

$$ 1 - 7Y^2 = 1 - 7(9k^2 + 6kr + r^2) = 1 - 63k^2 - 42kr - 7r^2 $$

Dividing by 3:

$$ X = \frac{1 - 63k^2 - 42kr - 7r^2}{3} $$

Simplify numerator:

$$ X = \frac{1}{3} - 21k^2 - 14kr - \frac{7r^2}{3} $$

Since $r \in \{1, 2\}$, $7r^2$ is either:

- For $r=1$: $7 \times 1 = 7$
- For $r=2$: $7 \times 4 = 28$

Now, check whether X is integer:

- For $r=1$:

$$ X = \frac{1}{3} - 21k^2 -14k - \frac{7}{3} = \left(\frac{1}{3} - \frac{7}{3}\right) - 21k^2 -14k = -\frac{6}{3} - 21k^2 -14k = -2 - 21k^2 -14k $$

which is an integer for all integer k.

- For $r=2$:

$$ X = \frac{1}{3} - 21k^2 -14k - \frac{28}{3} = \left(\frac{1}{3} - \frac{28}{3}\right) - 21k^2 -14k = -\frac{27}{3} - 21k^2 -14k = -9 - 21k^2 -14k $$

which is also an integer for all integer k.

Final formulas for solutions:

- When $Y \equiv 1 \pmod{3}$:

$$ Y = 3k + 1 $$
$$ X = -2 - 21k^2 - 14k $$

- When $Y \equiv 2 \pmod{3}$:

$$ Y = 3k + 2 $$
$$ X = -9 - 21k^2 - 14k $$

for all integers k.

Summary of All Solutions

The set of solutions (X, Y) in integers is:

$$
\boxed{
\begin{cases}
Y = 3k + 1, \quad X = -2 - 21k^2 - 14k \\
Y = 3k + 2, \quad X = -9 - 21k^2 - 14k
\end{cases}
\quad \text{where } k \in \mathbb{Z}
}
$$

This parametrization captures all solutions because:

- The divisibility condition ensures X is integral.
- The quadratic form of Y encompasses all integers satisfying the modular conditions.
- The solutions are infinite, indexed by k.

Examples of Solutions

Let's compute some specific solutions by choosing small values of k:

For $k=0$:

- $Y=1$:

$$ X = -2 - 0 - 0 = -2 $$
Solution: $(X, Y) = (-2, 1)$.

- $Y=2$:

$$ X = -9 - 0 - 0 = -9 $$
Solution: $(X, Y) = (-9, 2)$.

For $k=1$:

- \(Y=4\):

\[X = -2 - 21 - 14 = -37 \]
Solution: \((X, Y) = (-37, 4)\).

- \(Y=5\):

\[X = -9 - 21 - 14 = -44 \]
Solution: \((X, Y) = (-44, 5)\).

These examples confirm the pattern and demonstrate how solutions extend infinitely.

Conclusion and Justification

In conclusion, the solutions to the equation \(3X + 7Y^2 = 1\) are characterized by the parametric forms:

\[
\boxed{
\begin{cases}
Y = 3k + 1, \quad X = -2 - 21k^2 - 14k \\
Y = 3k + 2, \quad X = -9 - 21k^2 - 14k

Frequently Asked Questions

What is the main goal when solving the equation 3x + 7y^2 = 1 for integer solutions?

The main goal is to find all pairs of integers (x, y) that satisfy the equation, meaning both x and y are integers such that substituting them into the equation makes it true.

How can we determine if integer solutions exist for the equation 3x + 7y^2 = 1?

We can analyze the equation modulo 3 or 7 to check for divisibility conditions and use known properties of quadratic residues to determine the existence of solutions.

What method can be used to find all solutions to 3x + 7y^2 = 1?

One approach is to treat the equation as a Diophantine equation and attempt to solve for y^2 in terms of x, then analyze possible integer values of y by examining the equation modulo 7 or using substitution to identify potential solutions.

Are there any constraints on y for the equation 3x + 7y^2 = 1 to have solutions?

Yes, since 7y^2 appears in the equation, y must be an integer such that 7y^2 is close enough to 1 minus a multiple of 3, which limits possible y values to those where 7y^2 is congruent to 1 modulo 3.

What are the integer solutions (x, y) to the equation 3x + 7y^2 = 1?

The solutions are y = 0, which gives x = 1/3 (not integer), so no solutions with y=0. For y ≠ 0, analyzing the equation modulo 7 and 3 shows no integer y satisfies the conditions, so the only possible solution is when y=0, but x would not be integer. Therefore, there are no integer solutions to the equation.

Additional Resources

Number Theory: Find All Integer Solutions X, Y Such That $3X + 7Y^2 = 1$. Justify Your Answer!

Number theory, a fundamental branch of mathematics, often deals with the solutions of equations involving integers. These solutions, known as integer solutions or solutions over the integers, are vital for understanding the structure of numbers and their properties. One intriguing problem in this realm is to find all integer solutions \(X, Y\) to the equation:

$3X + 7Y^2 = 1$

This problem exemplifies the beauty and complexity of Diophantine equations—equations where solutions are sought within the set of integers. In this guide, we will explore how to approach this problem step-by-step, justify our findings with rigorous reasoning, and understand the nature of the solutions.

Understanding the Equation

The given equation:

\[3X + 7Y^2 = 1 \]

is a linear Diophantine equation in terms of \(X\) and \(Y\), with a quadratic term in \(Y\). Our goal is to find all integer pairs \((X, Y)\) that satisfy this equation.

Key observations:

- Since \(Y^2\) appears, and squares are non-negative, \(7Y^2\) is always non-negative.
- The term \(3X\) can be positive, negative, or zero, depending on \(X\).
- The right side is 1, which constrains the possible values of \(Y\).

Step 1: Isolate \(X\) and Consider Divisibility

Rearranging the original equation:

\[3X = 1 - 7Y^2 \]

Thus,

\[X = \frac{1 - 7Y^2}{3} \]

For \(X\) to be an integer, the numerator must be divisible by 3:

\[3 \mid (1 - 7Y^2) \]

which is equivalent to:

\[1 - 7Y^2 \equiv 0 \pmod{3} \]

Step 2: Modular Analysis to Find Possible \(Y\)

Let's analyze the divisibility condition modulo 3:

\[
1 - 7Y^2 \equiv 0 \pmod{3}
\]

Note that:

\[
7 \equiv 1 \pmod{3}
\]

since $7 = 3 \times 2 + 1$.

Therefore:

$$
1 - (1) \times Y^2 \equiv 0 \pmod{3}
$$

which simplifies to:

$$
1 - Y^2 \equiv 0 \pmod{3}
$$

or:

$$
Y^2 \equiv 1 \pmod{3}
$$

Now, quadratic residues modulo 3 are:

- $Y^2 \equiv 0 \pmod{3}$ if $Y \equiv 0 \pmod{3}$,
- $Y^2 \equiv 1 \pmod{3}$ if $Y \equiv \pm 1 \pmod{3}$.

Thus, the solutions for Y satisfy:

$$
Y \equiv \pm 1 \pmod{3}
$$

or explicitly,

$$
Y \equiv 1 \pmod{3} \quad \text{or} \quad Y \equiv 2 \pmod{3}
$$

since $2 \equiv -1 \pmod{3}$.

Step 3: Expressing X in Terms of Y

Recall:

\[
X = \frac{1 - 7Y^2}{3}
\]

Given the divisibility condition, \(X\) will be integer when \(Y\) satisfies the above congruences.

Now, for each possible residue class of \(Y\), we can examine whether \(X\) is integral and find the corresponding solutions.

Step 4: Find Explicit Solutions for \(Y\)

Case 1: \(Y \equiv 1 \pmod{3}\)

Let:

\[
Y = 3k + 1, \quad k \in \mathbb{Z}
\]

Calculate \(Y^2\):

\[
Y^2 = (3k + 1)^2 = 9k^2 + 6k + 1
\]

Then:

\[
7Y^2 = 7(9k^2 + 6k + 1) = 63k^2 + 42k + 7
\]

Now, compute numerator:

\[
1 - 7Y^2 = 1 - (63k^2 + 42k + 7) = -63k^2 - 42k - 6
\]

Divide by 3:

\[
X = \frac{-63k^2 - 42k - 6}{3} = -21k^2 - 14k - 2
\]

Since \(k \in \mathbb{Z}\), \(X\) is an integer for all integer \(k\).

Therefore, the solutions in this case are:

\[
\boxed{
\begin{cases}
Y = 3k + 1 \\
X = -21k^2 - 14k - 2
\end{cases}
\quad \text{for all } k \in \mathbb{Z}
}
\]

Case 2: \(Y \equiv 2 \pmod{3}\)

Let:

\[
Y = 3k + 2, \quad k \in \mathbb{Z}
\]

Calculate \(Y^2\):

\[
Y^2 = (3k + 2)^2 = 9k^2 + 12k + 4
\]

Then:

\[
7Y^2 = 7(9k^2 + 12k + 4) = 63k^2 + 84k + 28
\]

Calculate numerator:

\[

1 - 7Y^2 = 1 - (63k^2 + 84k + 28) = -63k^2 - 84k - 27
\]

Divide by 3:

\[
X = \frac{-63k^2 - 84k - 27}{3} = -21k^2 - 28k - 9
\]

Again, for all integer \(k\), \(X\) is an integer.

Thus, the solutions in this case are:

\[
\boxed{
\begin{cases}
Y = 3k + 2 \\
X = -21k^2 - 28k - 9
\end{cases}
\quad \text{for all } k \in \mathbb{Z}
}
\]

Summary of All Solutions

Combining both cases, the set of all integer solutions \((X, Y)\) to the original equation:

\[3X + 7Y^2 = 1 \]

is given by:

\[
\boxed{
\begin{aligned}
& \text{For all } k \in \mathbb{Z}: \\
& Y = 3k + 1, \quad X = -21k^2 - 14k - 2 \\
& \text{or} \\
& Y = 3k + 2, \quad X = -21k^2 - 28k - 9
\end{aligned}
}
\]

These parametric formulas generate all integer solutions.

Justification and Verification

Verifying the solutions:

- For any $k \in \mathbb{Z}$, substituting the formulas back into the original equation:

$$
3X + 7Y^2 = 1
$$

- For the first case:

$$
X = -21k^2 - 14k - 2, \quad Y = 3k + 1
$$

Calculate $3X + 7Y^2$:

$$
3(-21k^2 - 14k - 2) + 7(3k + 1)^2
$$

$$
= -63k^2 - 42k - 6 + 7(9k^2 + 6k + 1)
$$

$$
= -63k^2 - 42k - 6 + 63k^2 + 42k + 7
$$

$$
= (-63k^2 + 63k^2) + (-42k + 42k) + (-6 + 7) = 0 + 0 + 1 = 1
$$

- Similarly, for the second case:

$$
X = -21k^2 - 28k - 9, \quad Y = 3k + 2
$$

Calculate:

\[
3(-21k^2 - 28k - 9) + 7(3k + 2)^2
\]

\[
= -63k^2 - 84k - 27 + 7(9k^2 + 12k + 4)
\]

\[
= -63k^2 - 84k - 27 + 63k^2 + 84k + 28
\]

\[
= (-

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