Solve The Difference Equation Yx+2 + 4yx+1 + 3yx = 3* , Yo = 0, Y1 = 1 Using Z-transforms

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The process of solving difference equations is fundamental in discrete-time signal processing, control systems, and many areas of engineering and mathematics. Among the various techniques available, the Z-transform stands out as a powerful tool to handle linear difference equations with constant coefficients efficiently. In this comprehensive guide, we will explore how to solve the given difference equation using the Z-transform method, including step-by-step procedures, theoretical foundations, and practical insights.

Understanding the Difference Equation

Before delving into the solution, it's essential to understand the structure of the given difference equation and the initial conditions. The equation is:

$$ y_{x+2} + 4 y_{x+1} + 3 y_{x} = 3^x $$

with initial conditions:

$$ y_0 = 0, \quad y_1 = 1 $$

This is a nonhomogeneous linear difference equation of second order with constant coefficients, driven by an exponential input 3^x. Our goal is to find a general solution for y_x that satisfies both the equation and the initial conditions.

Fundamentals of Z-Transform

The Z-transform converts discrete-time difference equations into algebraic equations in the Z-domain, simplifying the process of solving for the sequence y_x.

Definition of Z-Transform

The Z-transform of a sequence y_x is defined as:

$$Y(z) = \mathcal{Z}\{ y_x \} = \sum_{x=0}^{\infty} y_x z^{-x}$$

where z is a complex variable.

Key Properties

- Linearity: $\mathcal{Z}\{ a y_x + b z_x \} = a Y(z) + b Z(z)$
- Shifting in x:
$$\mathcal{Z}\{ y_{x+k} \} = z^{k} [Y(z) - y_0 z^{-1} - y_1 z^{-2} - \dots - y_{k-1} z^{-(k)}]$$
- Initial value theorem: $y_0 = \lim_{z \to \infty} Y(z)$ (for causal sequences)

Applying the Z-Transform to the Difference Equation

To solve the difference equation, we follow these steps:

1. Take the Z-transform of both sides of the equation.
2. Express the transformed equation in terms of $Y(z)$.
3. Incorporate initial conditions into the Z-transform.
4. Solve for $Y(z)$.
5. Find the inverse Z-transform to obtain y_x.

Step 1: Z-Transform of the Left-Hand Side

Using the shifting property:

$$\mathcal{Z}\{ y_{x+2} \} = z^{2} [Y(z) - y_0 z^{-1} - y_1]$$
$$\mathcal{Z}\{ y_{x+1} \} = z [Y(z) - y_0]$$
$$\mathcal{Z}\{ y_{x} \} = Y(z)$$

Substituting these into the original equation:

$$z^{2} [Y(z) - y_0 z^{-1} - y_1] + 4 z [Y(z) - y_0] + 3 Y(z) = \mathcal{Z}\{ 3^x \}$$

Note that the right side is the Z-transform of 3^x.

Step 2: Z-Transform of the Input 3^x

The Z-transform of an exponential sequence a^x is:

$$\mathcal{Z}\{ a^x \} = \frac{z}{z - a}$$

for $|z| > |a|$.

Thus,

$$\mathcal{Z}\{ 3^x \} = \frac{z}{z - 3}$$

Step 3: Substitute and Simplify

Substituting initial conditions $y_0 = 0$, $y_1 = 1$:

$$z^{2} [Y(z) - 0 \cdot z^{-1} - 1] + 4 z [Y(z) - 0] + 3 Y(z) = \frac{z}{z - 3}$$

Simplify:

$$z^{2} (Y(z) - 1) + 4 z Y(z) + 3 Y(z) = \frac{z}{z - 3}$$

Expanding:

$$z^{2} Y(z) - z^{2} + 4 z Y(z) + 3 Y(z) = \frac{z}{z - 3}$$

Combine terms with $Y(z)$:

$$(z^{2} + 4 z + 3) Y(z) = \frac{z}{z - 3} + z^{2}$$

Solving for $Y(z)$

Express $Y(z)$:

$$Y(z) = \frac{\frac{z}{z - 3} + z^{2}}{z^{2} + 4 z + 3}$$

Combine the numerator over a common denominator:

$$ Y(z) = \frac{\frac{z + z^{2}(z - 3)}{z - 3}}{(z + 1)(z + 3)} $$

Note that $z^2 + 4z + 3 = (z + 1)(z + 3)$.

Simplify numerator:

$$ z + z^{2}(z - 3) = z + z^{3} - 3 z^{2} $$

So,

$$ Y(z) = \frac{\frac{z + z^{3} - 3 z^{2}}{z - 3}}{(z + 1)(z + 3)} $$

Rewrite as:

$$ Y(z) = \frac{z + z^{3} - 3 z^{2}}{(z - 3)(z + 1)(z + 3)} $$

Inverse Z-Transform and Final Solution

The next step is to perform partial fraction decomposition to simplify $Y(z)$ and then find the inverse Z-transform to obtain the explicit form of y_x.

Partial Fraction Decomposition

Express:

$$ \frac{z + z^{3} - 3 z^{2}}{(z - 3)(z + 1)(z + 3)} = \frac{A}{z - 3} + \frac{B}{z + 1} + \frac{C}{z + 3} $$

Multiply both sides by the denominator:

$$ z + z^{3} - 3 z^{2} = A (z + 1)(z + 3) + B (z - 3)(z + 3) + C (z - 3)(z + 1) $$

Calculate each term:

- $A (z + 1)(z + 3) = A (z^2 + 4z + 3)$
- $B (z - 3)(z + 3) = B (z^2 - 9)$
- $C (z - 3)(z + 1) = C (z^2 - 2z - 3)$

Now, combine the right side:

$$ A z^2 + 4 A z + 3A + B z^2 - 9 B + C z^2 - 2 C z - 3 C $$

Group like terms:

$$ (A + B + C) z^2 + (4 A - 2 C) z + (3A - 9 B - 3 C) $$

Set equal to the left side:

$$ z + z^3 - 3 z^2 $$

Note that the highest degree term on the left is z^3, but on the right, the highest degree term is z^2. To match powers, observe that the numerator of $Y(z)$ is degree 3, so the partial fraction expansion might need adjustments. Alternatively, perform polynomial division or factor numerator to facilitate inverse transform.

Alternatively, observe the numerator:

$$ z + z^{3} - 3 z^{2} = z^{3} - 3 z^{2} + z $$

Express $Y(z)$ as:

\[Y(z) = \frac{z^{3} -

Frequently Asked Questions

What is the main goal when solving the difference equation Yx+2 + 4yx+1 + 3yx = 3 using Z-transforms?

The main goal is to find the explicit formula for the sequence Yx by applying Z-transforms to convert the difference equation into an algebraic equation, then solving for Y(z) and performing the inverse Z-transform.

How do initial conditions Yo=0 and Y1=1 influence the Z-transform solution of the difference equation?

Initial conditions are used to determine the constants when performing the inverse Z-transform, ensuring the particular solution satisfies the initial values Yo=0 and Y1=1.

What is the Z-transform of the difference equation Yx+2 + 4yx+1 + 3yx = 3?

Applying Z-transform to each term, the equation becomes Z{Yx+2} + 4Z{Yx+1} + 3Z{Yx} = Z{3}. Using properties of Z-transforms, this translates into a rational algebraic equation in terms of Y(z).

How do you handle the shift terms, such as Yx+1 and

Yx+2, in the Z-transform method?

Shifts in the sequence correspond to multiplication by powers of Z in the Z-domain: Z{Yx+1} = ZY(z) - Y(0), and Z{Yx+2} = Z^2Y(z) - ZY(0) - Y(1).

What is the step-by-step process to find Y(z) for the given difference equation?

First, take the Z-transform of each term, substitute initial conditions, solve the resulting algebraic equation for Y(z), and then find the inverse Z-transform to get Yx.

How do the initial conditions Yo=0 and Y1=1 affect the algebraic solution for Y(z)?

They are used to evaluate the transformed equation, specifically replacing Y(0) and Y(1) terms, which simplifies the expression for Y(z).

What techniques are used to perform the inverse Z-transform to find the explicit formula for Yx?

Partial fraction decomposition, power series expansion, or leveraging known Z-transform pairs are used to invert Y(z) back to the sequence Yx.

What is the significance of the stability and causality conditions in solving this difference equation via Z-transforms?

They ensure the solution is physically meaningful and converges, guiding the choice of the region of convergence when performing the inverse Z-transform.

Are there any common challenges or pitfalls when solving difference equations with Z-transforms, such as in this example?

Yes, common challenges include correctly handling initial conditions, choosing the proper region of convergence, and accurately performing the inverse Z-transform, especially for complex rational functions.

Additional Resources

Solve The Difference Equation Yx+2 + 4yx+1 + 3yx = 3, Yo = 0, Y1 = 1 Using Z-transforms is an intriguing problem that combines the fundamentals of difference equations with the powerful tool of Z-transforms. This approach is particularly useful for solving linear, constant-coefficient difference equations, especially when initial conditions are specified. In this comprehensive guide, we will explore step-by-step how to approach, manipulate, and solve such difference equations using Z-transforms, providing clarity for students, engineers, or anyone interested in discrete-time systems analysis.

Introduction to Difference Equations and Z-Transforms

Difference equations describe the relationship between the values of a sequence at different points. They are discrete-time analogs of differential equations and are fundamental in digital signal processing, control systems, and various engineering fields.

The Z-transform is a powerful mathematical tool that converts difference equations from the time domain into the complex frequency domain, simplifying algebraic manipulations and solution processes. Once the algebraic solution is obtained in the Z-domain, the inverse Z-transform retrieves the sequence in the time domain.

Understanding the Given Problem

Let's restate the problem for clarity:

- Difference Equation:
$y_{x+2} + 4 y_{x+1} + 3 y_x = 3$

- Initial Conditions:
$y_0 = 0$
$y_1 = 1$

Our goal is to find the explicit expression for y_x using Z-transforms.

Step 1: Recognize the Form of the Difference Equation

The given equation is a second-order linear difference equation with constant coefficients and a constant non-homogeneous term:

$$
y_{x+2} + 4 y_{x+1} + 3 y_x = 3
$$

The initial conditions are specified for y_0 and y_1.

Step 2: Applying the Z-Transform

What is the Z-Transform?

The Z-transform of a sequence y_x, denoted $Y(z)$, is defined as:

$$
Y(z) = \sum_{x=0}^{\infty} y_x z^{-x}
$$

Transforming the Difference Equation

Applying the Z-transform to each term:

- $Z\{ y_{x+2} \} = z^2 (Y(z) - y_0 - y_1 z^{-1})$
- $Z\{ y_{x+1} \} = z (Y(z) - y_0)$
- $Z\{ y_x \} = Y(z)$

The constant term 3 transforms to:

$$
Z\{ 3 \} = \frac{3 z}{z - 1}
$$

Putting it all together:

$$
z^2 (Y(z) - y_0 - y_1 z^{-1}) + 4 z (Y(z) - y_0) + 3 Y(z) = \frac{3 z}{z - 1}
$$

Step 3: Substituting Initial Conditions

Using $y_0 = 0$, $y_1 = 1$:

$$
z^2 (Y(z) - 0 - 1 \cdot z^{-1}) + 4 z (Y(z) - 0) + 3 Y(z) = \frac{3 z}{z - 1}
$$

Simplify:

$$
z^2 (Y(z) - z^{-1}) + 4 z Y(z) + 3 Y(z) = \frac{3 z}{z - 1}
$$

$$
z^2 Y(z) - z^2 z^{-1} + 4 z Y(z) + 3 Y(z) = \frac{3 z}{z - 1}
$$

Recall that $z^2 z^{-1} = z$, so:

$$
z^2 Y(z) - z + 4 z Y(z) + 3 Y(z) = \frac{3 z}{z - 1}
$$

Group the $Y(z)$ terms:

$$
(z^2 + 4 z + 3) Y(z) = \frac{3 z}{z - 1} + z
$$

Step 4: Solving for $Y(z)$

Express $Y(z)$:

$$
Y(z) = \frac{\frac{3 z}{z - 1} + z}{z^2 + 4 z + 3}
$$

Combine the numerator:

$$
Y(z) = \frac{\frac{3 z + z (z - 1)}{z - 1}}{(z + 1)(z + 3)}
$$

Note that $z^2 + 4 z + 3 = (z + 1)(z + 3)$. Now, simplify numerator:

$$
3 z + z (z - 1) = 3 z + z^2 - z = z^2 + 2 z
$$

So:

$$
Y(z) = \frac{\frac{z^2 + 2 z}{z - 1}}{(z + 1)(z + 3)} = \frac{z^2 + 2 z}{(z - 1)(z + 1)(z + 3)}
$$

Step 5: Partial Fraction Expansion

To perform the inverse Z-transform, decompose $Y(z)$ into simpler fractions:

$$
Y(z) = \frac{z^2 + 2 z}{(z - 1)(z + 1)(z + 3)}
$$

Express as:

$$
Y(z) = \frac{A}{z - 1} + \frac{B}{z + 1} + \frac{C}{z + 3}
$$

Multiply both sides by the denominator:

$$
z^2 + 2 z = A (z + 1)(z + 3) + B (z - 1)(z + 3) + C (z - 1)(z + 1)
$$

Calculate each:

- $(z + 1)(z + 3) = z^2 + 4 z + 3$
- $(z - 1)(z + 3) = z^2 + 2 z - 3$
- $(z - 1)(z + 1) = z^2 - 1$

Rewrite:

$$
z^2 + 2 z = A (z^2 + 4 z + 3) + B (z^2 + 2 z - 3) + C (z^2 - 1)
$$

Group coefficients:

$$
z^2 + 2 z = (A + B + C) z^2 + (4 A + 2 B) z + (3 A - 3 B - C)
$$

Matching coefficients:

- For z^2:

$$
1 = A + B + C
$$

- For z:

$$
2 = 4 A + 2 B
$$

- For constant term:

$$
0 = 3 A - 3 B - C
$$

Step 6: Solving for Coefficients

From the z coefficient:

$$
2 = 4 A + 2 B \Rightarrow 1 = 2 A + B \Rightarrow B = 1 - 2 A
$$

From the constant term:

$$
0 = 3 A - 3 B - C \Rightarrow C = 3 A - 3 B
$$

Substitute B:

$$
C = 3 A - 3 (1 - 2 A) = 3 A - 3 + 6 A = 9 A - 3
$$

From the z^2 coefficient:

$$
1 = A + B + C
$$

Substitute B and C:

$$
1 = A + (1 - 2 A) + (9 A - 3) = A + 1 - 2 A + 9 A - 3 = (A - 2 A + 9 A) + (1 - 3) = (8 A) - 2
$$

Solve for A:

$$
8 A = 3 \Rightarrow A = \frac{3}{8}
$$

Now, find B:

$$
B = 1 - 2 \times \frac{3}{8} = 1 - \frac{6}{8} = 1 - \frac{3}{4} = \frac{1}{4}
$$

Find C:

$$
C = 9 \times \frac{3}{8} - 3 = \frac{27}{8} - \frac{24}{8} = \frac{3}{8}
$$

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Step 7: Final Expression in the Z-Domain

Now, \( Y(z) \) is expressed as:

\[
Y(z) = \frac{A}{z - 1} + \frac{B}{z + 1} + \frac{C}{z + 3}
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Solve The Difference Equation Yx 2 4yx 1 3yx 3 Yo 0 Y1 1 Using Z Transforms

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