Solve The Initial Value Problem Dx/dt = Ax With X(0) = Xo- -1 -2 1 A= Xo = -[5] 2 -2 5 X(t)

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Introduction to the Initial Value Problem (IVP)

When dealing with systems of differential equations, especially linear systems, understanding how to solve an initial value problem (IVP) is essential. The specific problem presented involves a first-order linear differential equation of the form:

$$\frac{dX}{dt} = A X$$

where:

- $X(t)$ is a vector function representing the state variables over time,
- A is a constant matrix defining the system dynamics,
- $X(0) = X_0$ is the initial condition vector.

In this case, the problem is explicitly given as:

$$\frac{dX}{dt} = A X$$

with initial condition:

$$X(0) = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

and the matrix:

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 5 & 2 & -2 \\ 5 & 2 & -2 \end{bmatrix}$$

Note that the matrix A appears to have repeated rows, indicating a potential pattern or structure that might influence the solution method.

Understanding the System Components

The Differential Equation

The system is a set of coupled first-order linear differential equations, represented in matrix form as:

$$\frac{dX}{dt} = A X$$

where:

- $X(t)$ is a vector in $\mathbb{R}^3$,
- A is a 3×3 constant matrix.

Initial Conditions

The initial state vector is given as:

$$X_0 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

These initial conditions are essential for determining the specific solution to the IVP.

The System Matrix A

Given as:

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 5 & 2 & -2 \\ 5 & 2 & -2 \end{bmatrix}$$

This matrix has identical rows, which simplifies the analysis by indicating the rank and eigenvalues.

Step-by-Step Solution Approach

To solve this initial value problem, follow these core steps:

1. Find the Eigenvalues of A
2. Compute the Eigenvectors associated with each eigenvalue
3. Construct the general solution using eigenvalues and eigenvectors
4. Apply initial conditions to find the specific solution

Step 1: Find Eigenvalues of A

Eigenvalues are solutions to the characteristic equation:

$$\det(A - \lambda I) = 0$$

where I is the identity matrix.

Given the matrix A:

$$
A - \lambda I =
\begin{bmatrix}
5 - \lambda & 2 & -2 \\
5 & 2 - \lambda & -2 \\
5 & 2 & -2 - \lambda
\end{bmatrix}
$$

Calculating the determinant:

$$
\det(A - \lambda I) = 0
$$

Because the rows are identical in the first two rows but the third row differs, the calculation simplifies by row operations or direct expansion.

Observation: Since the first two rows are identical, the matrix is rank deficient, which suggests at least one eigenvalue is associated with the repeated row structure.

Calculating the eigenvalues:

- The sum of each row:

$$
R_1: 5 + 2 + (-2) = 5
$$

Since rows 1 and 2 are identical, their sum is 5, indicating that $\lambda = 5$ is an eigenvalue.

- To find the other eigenvalues, perform the characteristic polynomial calculation explicitly.

Characteristic polynomial:

$$
\det(A - \lambda I) = (5 - \lambda) \det \begin{bmatrix} 2 - \lambda & -2 \\ 2 & -2 - \lambda \end{bmatrix} - 2 \det \begin{bmatrix} 5 & -2 \\ 5 & -2 - \lambda \end{bmatrix} + (-2) \det \begin{bmatrix} 5 & 2 - \lambda \\ 5 & 2 \end{bmatrix}
$$

Calculating these determinants:

- First minor:

$$
(2 - \lambda)(-2 - \lambda) - (-2)(2) = [(2 - \lambda)(-2 - \lambda)] + 4
$$

- Second minor:

$$
5(-2 - \lambda) - (-2)(5) = -5(\lambda + 2) + 10
$$

- Third minor:

$$
5(2) - (2 - \lambda)(5) = 10 - 5(2 - \lambda) = 10 - (10 - 5\lambda) = 5 \lambda
$$

Putting these together:

$$
\det(A - \lambda I) = (5 - \lambda) [(2 - \lambda)(-2 - \lambda) + 4] - 2 [-5(\lambda + 2) + 10] + (-2)(5 \lambda)
$$

Simplify step-by-step:

- Compute $(2 - \lambda)(-2 - \lambda)$:

$$
(2 - \lambda)(-2 - \lambda) = 2 \times -2 + 2 \times -\lambda - \lambda \times -2 - \lambda \times -\lambda = -4 - 2 \lambda + 2 \lambda + \lambda^2 = -4 + \lambda^2
$$

Adding 4:

$$
-4 + \lambda^2 + 4 = \lambda^2
$$

Now, the determinant:

$$
\det(A - \lambda I) = (5 - \lambda) \times \lambda^2 - 2 [-5 (\lambda + 2) + 10] - 2 \times 5 \lambda
$$

Simplify each term:

- First term:

$$
(5 - \lambda) \lambda^2 = 5 \lambda^2 - \lambda^3
$$

- Second term:

$$
-2 [-5 (\lambda + 2) + 10] = -2 [-5 \lambda - 10 + 10] = -2 (-5 \lambda) = 10 \lambda
$$

- Third term:

$$
-2 \times 5 \lambda = -10 \lambda
$$

Adding all together:

\[

\det(A - \lambda I) = 5 \lambda^2 - \lambda^3 + 10 \lambda - 10 \lambda = 5 \lambda^2 - \lambda^3
\]

Simplify:

\[
\det(A - \lambda I) = -\lambda^3 + 5 \lambda^2
\]

Factor:

\[
-\lambda^2 (\lambda - 5) = 0
\]

Eigenvalues:

\[
\boxed{
\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 5
}
\]

Summary of Eigenvalues:

- Eigenvalue 0 with multiplicity 2
- Eigenvalue 5 with multiplicity 1

Step 2: Find Eigenvectors

Eigenvector for \(\lambda = 5 \):

Solve:

\[
(A - 5 I) X = 0
\]

\[
A - 5 I =
\begin{bmatrix}
0 & 2 & -2 \\
5 & -3 & -2 \\
5 & 2 & -7
\end{bmatrix}
\]

From the first row:

\[
0 \times x_1 + 2 x_2 - 2 x_3 = 0 \Rightarrow 2 x_2 = 2 x_3 \Rightarrow x_2 = x_3
\]

From the second row:

\[
5 x_1 - 3 x_2 - 2 x_3 = 0
\]

Substitute \(x_2 = x_3 \):

\[
5 x_1 - 3 x_3 - 2 x_3 = 0 \Rightarrow 5 x_1 - 5 x_3 = 0 \Rightarrow x_1 = x_3
\]

Set \(x_3 = t \), then:

\[
x_1 = t, \quad x_2 = t, \quad x_3 = t
\]

Eigenvector:

\[
X^{(5)} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

Eigenvectors for \(\lambda = 0 \):

Solve:

\[
A X = 0
\]

\[
\begin{bmatrix}
5 & 2 & -2 \\
5 & 2 & -2 \\
5 & 2 & -2
\end{bmatrix

Frequently Asked Questions

What is the initial value problem given in the differential equation Dx/dt = A x?

The initial value problem is Dx/dt = A x with the initial condition X(0) = X0, where A = [[-1, -2, 1], [2, -2, 5]] and X0 = [-5, 2, -2].

How do you solve the differential equation Dx/dt = A x with the given initial condition?

You solve it by computing the matrix exponential e^{A t} and then multiplying by the initial vector X0: X(t) = e^{A t} X0.

What is the significance of matrix A in solving this initial value problem?

Matrix A determines the system's dynamics; its eigenvalues and eigenvectors are used to compute the matrix exponential, which provides the solution over time.

How can the eigenvalues and eigenvectors of matrix A help in solving the IVP?

Eigenvalues and eigenvectors facilitate diagonalizing A or finding its Jordan form, simplifying the computation of e^{A t} and thus solving the system explicitly.

What is the general solution form for the system Dx/dt = A x?

The general solution is X(t) = e^{A t} X0, where e^{A t} is the matrix exponential of A times t.

Are there any specific challenges in solving this IVP with the given matrix A?

Yes, if A has complex eigenvalues or is not diagonalizable, computing e^{A t} becomes more involved, requiring Jordan form or other methods.

Can this system be solved analytically, and what tools are typically used?

Yes, it can be solved analytically by diagonalization or Jordan form; computational tools like MATLAB, WolframAlpha, or Python's SciPy can assist with matrix exponential calculations.

What physical or real-world systems can this type of differential equation model?

Such systems model coupled oscillations, electrical circuits, population dynamics, or any phenomena described by linear systems of differential equations.

Additional Resources

Solve The Initial Value Problem Dx/dt = Ax With X(0) = Xo- -1 -2 1 A= Xo = -[5] 2 -2 5 X(t)

Solving initial value problems (IVPs) for systems of differential equations is a fundamental aspect of applied mathematics, engineering, physics, and many other scientific disciplines. The specific problem presented involves a system of first-order linear differential equations with an initial condition, represented succinctly as:

$$\frac{dX}{dt} = A X, \quad \text{with} \quad X(0) = X_0.$$

In this particular case, the matrix A and the initial vector X_0 are given as:

$$
A = \begin{bmatrix}
-5 & 2 \\
-2 & 5
\end{bmatrix}, \quad
X_0 = \begin{bmatrix}
-1 \\
-2
\end{bmatrix}.
$$

This problem exemplifies the classic approach to solving linear systems of differential equations using matrix exponentials, eigenvalues, and eigenvectors. In this review, we will explore the solution methodology step-by-step, analyze the properties of the matrix A, discuss the implications of the initial conditions, and evaluate the overall approach with its advantages and limitations.

Understanding the Problem and Its Significance

The differential equation system:

$$
\frac{dX}{dt} = A X,
$$
with initial condition:

$$
X(0) = X_0,
$$
is a standard form for modeling dynamic systems where the rate of change of the state vector $X(t)$ depends linearly on $X(t)$. Examples include mechanical systems, electrical circuits, population models, and economic systems.

Key features of this problem:
- It involves a constant coefficient matrix A, which simplifies the solution process.

- The initial condition specifies the system's starting point, critical for obtaining a unique solution.
- The matrix A is real and symmetric, with eigenvalues likely complex conjugates, indicating potential oscillatory behavior.

Understanding how to solve such systems provides insights into system stability, long-term behavior, and response characteristics.

Mathematical Approach to Solving the System

1. General Solution Framework

The general solution to a system:

$$
\frac{dX}{dt} = A X,
$$
is given by:

$$
X(t) = e^{A t} X_0,
$$
where $e^{A t}$ is the matrix exponential of $A t$. Computing this matrix exponential is central to solving the system.

---

2. Computing the Matrix Exponential

The matrix exponential can be computed via:
- Eigen-decomposition: If A is diagonalizable, write $A = P D P^{-1}$, where D is a diagonal matrix of eigenvalues, and P consists of eigenvectors. Then:

$$
e^{A t} = P e^{D t} P^{-1},
$$
with

$$
e^{D t} = \begin{bmatrix}
e^{\lambda_1 t} & 0 \\
0 & e^{\lambda_2 t}
\end{bmatrix}.
$$

- Jordan canonical form: If A is not diagonalizable, use Jordan form to compute the exponential.

- Series expansion or numerical methods: For more complex matrices, series or numerical algorithms like Padé approximants may be used.

In our case, eigen-decomposition is preferable due to the simplicity of A.

Eigenvalues and Eigenvectors of A

1. Finding Eigenvalues

Eigenvalues λ satisfy:

$$
\det(A - \lambda I) = 0,
$$
where I is the identity matrix.

Calculate:

$$
A - \lambda I = \begin{bmatrix}
-5 - \lambda & 2 \\
-2 & 5 - \lambda
\end{bmatrix}.
$$

Determinant:

$$
(-5 - \lambda)(5 - \lambda) - (-2)(2) = 0,
$$
which simplifies to:

$$
(-5 - \lambda)(5 - \lambda) + 4 = 0.
$$

Expanding:

$$
(-5)(5 - \lambda) - \lambda(5 - \lambda) + 4 = 0,
$$
$$
-25 + 5 \lambda - 5 \lambda + \lambda^2 + 4 = 0,

\]
\[
\lambda^2 - 21 = 0,
\]
so

\[
\lambda^2 = 21,
\]
\[
\lambda = \pm \sqrt{21} \approx \pm 4.58.
\]

Eigenvalues:

\[
\boxed{
\lambda_1 \approx 4.58, \quad \lambda_2 \approx -4.58.
}
\]

---

2. Eigenvectors Calculation

For \(\lambda_1 \approx 4.58\):

Solve:

\[
(A - \lambda_1 I) v = 0,
\]
which becomes:

\[
\begin{bmatrix}
-5 - 4.58 & 2 \\
-2 & 5 - 4.58
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = 0,
\]
or:

\[
\begin{bmatrix}
-9.58 & 2 \\

-2 & 0.42
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = 0.
\]

From the first row:

\[
-9.58 v_1 + 2 v_2 = 0 \Rightarrow v_2 = \frac{9.58}{2} v_1 \approx 4.79 v_1.
\]

Choosing \(v_1 = 1\), the eigenvector:

\[
v^{(1)} \approx \begin{bmatrix} 1 \\ 4.79 \end{bmatrix}.
\]

Similarly, for \(\lambda_2 \approx -4.58\):

\[
A - \lambda_2 I = \begin{bmatrix}
-5 + 4.58 & 2 \\
-2 & 5 + 4.58
\end{bmatrix} = \begin{bmatrix}
-0.42 & 2 \\
-2 & 9.58
\end{bmatrix}.
\]

From the first row:

\[
-0.42 v_1 + 2 v_2 = 0 \Rightarrow v_2 = \frac{0.42}{2} v_1 \approx 0.21 v_1.
\]

Choosing \(v_1=1\), eigenvector:

\[
v^{(2)} \approx \begin{bmatrix} 1 \\ 0.21 \end{bmatrix}.
\]

Constructing the Solution

With eigenvalues and eigenvectors computed, the general solution is:

$$
X(t) = c_1 e^{\lambda_1 t} v^{(1)} + c_2 e^{\lambda_2 t} v^{(2)}.
$$

Applying the initial condition $X(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$:

$$
X(0) = c_1 v^{(1)} + c_2 v^{(2)} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.
$$

Set up the system:

$$
\begin{bmatrix}
1 & 1 \\
4.79 & 0.21
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} =
\begin{bmatrix}
-1 \\
-2
\end{bmatrix}.
$$

Solve for c_1, c_2:

From the first equation:

$$
c_1 + c_2 = -1 \Rightarrow c_2 = -1 - c_1.
$$

Substitute into second:

$$
4.79 c_1 + 0.21 (-1 - c_1) = -2,
$$
$$
4.79 c_1 - 0.21 - 0.21 c_1 = -2,
$$
$$
(4.79 - 0.21) c_1 = -2 + 0.21,
$$
$$
4.58 c_1 = -1.79,
$$
$$
c_1 \approx -1.79 / 4.58 \approx -0.39.

\]

Then:

\[
c_2 = -1 - (-0.39) = -0.61.
\]

Final solution:

\[
X(t) = -0.39 e^{4.58 t} \begin{bmatrix} 1 \\ 4.79

Solve The Initial Value Problem Dx Dt Ax With X 0 Xo 1 2 1 A Xo 5 2 2 5 X T

Solve The Initial Value Problem Dx Dt Ax With X 0 Xo 1 2 1 A Xo 5 2 2 5 X T

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