What Is A Quartic Polynomial Function With Rational Coefficients And Roots Of 1,-1, And 4i?

What Is A Quartic Polynomial Function With Rational Coefficients And Roots Of 1, -1, And 4i?

A quartic polynomial function is a polynomial of degree four, meaning its highest power of the variable (usually denoted as x) is four. When such a polynomial has rational coefficients and specific roots—such as 1, -1, and 4i—it becomes a fascinating subject for algebra and polynomial theory. Understanding how to construct and analyze these functions involves grasping concepts like roots, coefficients, conjugate pairs, and polynomial expansion. This article explores what defines such functions, how their roots influence their form, and the methods to derive the polynomial explicitly.

Understanding Quartic Polynomial Functions

A quartic polynomial function is expressed generally as:

```plaintext
f(x) = ax^4 + bx^3 + cx^2 + dx + e
```

where:
- a, b, c, d, e are coefficients, with a ≠ 0.
- The degree of the polynomial is 4, indicating the highest exponent of x is 4.

These functions can have up to four roots (solutions), which may be real or complex. The Fundamental Theorem of Algebra states that any degree-four polynomial has exactly four roots in the complex plane, counting multiplicities.

Roots of a Quartic Polynomial With Rational Coefficients

When a polynomial has rational coefficients, its roots are interconnected through complex conjugation properties and polynomial symmetry.

Why Roots Come in Conjugate Pairs

- If the polynomial has rational coefficients and a complex root, then its complex conjugate must also be a root.
- For example, if 4i is a root, then -4i must also be a root because the coefficients are rational.

Implication for Roots of 1, -1, and 4i

Given roots:
- 1 (real root)
- -1 (real root)
- 4i (complex root)
- -4i (complex conjugate root of 4i)

The roots are symmetric with respect to the real axis, ensuring the polynomial has rational coefficients.

Constructing the Polynomial from Given Roots

When roots are known, the polynomial can be constructed by taking the product of factors corresponding to each root:

```plaintext
f(x) = a × (x - root_1) × (x - root_2) × (x - root_3) × (x - root_4)
```

Since the leading coefficient a can be any non-zero rational number, we typically choose a = 1 for simplicity.

Factors Based on Roots

Given roots 1, -1, 4i, and -4i, the factors are:

1. (x - 1)
2. (x + 1)
3. (x - 4i)
4. (x + 4i)

The polynomial becomes:

```plaintext
f(x) = (x - 1)(x + 1)(x - 4i)(x + 4i)
```

Expanding the Polynomial

To find a standard form, expand the factors step-by-step.

Step 1: Expand the real roots factors

```plaintext
(x - 1)(x + 1) = x^2 - 1
```

This is a difference of squares.

Step 2: Expand the complex roots factors

```plaintext
(x - 4i)(x + 4i) = x^2 - (4i)^2
```

Since:

```plaintext
(4i)^2 = 16i^2 = 16 × (-1) = -16
```

Therefore:

```plaintext
(x - 4i)(x + 4i) = x^2 - (-16) = x^2 + 16
```

Step 3: Multiply the two quadratics

Now, multiply:

```plaintext
(x^2 - 1)(x^2 + 16)
```

Applying the distributive property:

```plaintext
x^2 × x^2 = x^4
x^2 × 16 = 16x^2
-1 × x^2 = -x^2
-1 × 16 = -16
```

Combine like terms:

```plaintext
x^4 + (16x^2 - x^2) - 16 = x^4 + 15x^2 - 16
```

The Final Polynomial Equation

Thus, the quartic polynomial with rational coefficients and roots 1, -1, 4i, and -4i is:

```plaintext
f(x) = x^4 + 15x^2 - 16
```

This polynomial satisfies all the initial conditions:
- It is quartic.
- It has rational coefficients.
- It has the specified roots.

Verifying the Roots of the Polynomial

To confirm the roots, substitute each into the polynomial:

- For x = 1:

```plaintext
f(1) = 1^4 + 15(1)^2 - 16 = 1 + 15 - 16 = 0
```

- For x = -1:

```plaintext
f(-1) = (-1)^4 + 15(-1)^2 - 16 = 1 + 15 - 16 = 0
```

```

- For x = 4i:

```plaintext
f(4i) = (4i)^4 + 15(4i)^2 - 16
```

Calculate each term:

```plaintext
(4i)^2 = 16i^2 = -16
(4i)^4 = [(4i)^2]^2 = (-16)^2 = 256
```

Now, substitute:

```plaintext
f(4i) = 256 + 15(-16) - 16 = 256 - 240 - 16 = 0
```

Similarly, for x = -4i:

```plaintext
(-4i)^2 = 16i^2 = -16
(-4i)^4 = (-16)^2 = 256
```

Substitute:

```plaintext
f(-4i) = 256 + 15(-16) - 16 = 256 - 240 - 16 = 0
```

All roots satisfy the polynomial, confirming its correctness.

Additional Properties and Applications

Symmetry and Rational Coefficients

- The polynomial's symmetry reflects the roots' conjugate pairs.
- Rational coefficients ensure the polynomial’s behavior is predictable and applicable in various fields like algebra, number theory, and engineering.
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Applications of Such Polynomials

- Designing systems with specific oscillatory behaviors.
- Solving algebraic equations with complex roots.
- Analyzing stability in control systems.
- Understanding polynomial factorization in abstract algebra.

Conclusion

A quartic polynomial function with rational coefficients and roots of 1, -1, and 4i can be explicitly constructed by utilizing the roots to form factors, then expanding those factors into a standard polynomial form. The key steps involve recognizing the conjugate pair for the complex roots, expanding step-by-step, and verifying that the polynomial satisfies the initial root conditions. The resulting polynomial, $f(x) = x^4 + 15x^2 - 16$, exemplifies how algebraic principles intertwine roots, coefficients, and polynomial structure. Understanding such functions deepens comprehension of algebraic solutions and their applications across mathematics and engineering.

Frequently Asked Questions

What is a quartic polynomial function with rational coefficients that has roots 1, -1, and 4i?

A quartic polynomial with rational coefficients having roots 1, -1, and 4i must also include the conjugate root -4i to ensure rational coefficients. Its factored form is (x - 1)(x + 1)(x - 4i)(x + 4i).

How do roots 4i and -4i influence the coefficients of the quartic polynomial?

Since 4i and -4i are complex conjugates, their factors combine to produce a quadratic with real coefficients, specifically (x^2 + 16). This ensures the overall polynomial has rational coefficients.

What is the explicit form of the quartic polynomial with roots 1, -1, 4i, and -4i?

Multiplying the factors: (x - 1)(x + 1) = x^2 - 1, and (x - 4i)(x + 4i) = x^2 + 16. The quartic polynomial is (x^2 - 1)(x^2 + 16) = x^4 + 16x^2 - x^2 - 16 = x^4 + 15x^2 - 16.

Are all roots of the polynomial rational or complex in this case?

The roots are 1, -1, 4i, and -4i. The roots 1 and -1 are rational, while 4i and -4i are complex conjugates. The polynomial's coefficients are rational, but the roots include both rational and complex numbers.

Why must the polynomial include the conjugate root -4i when 4i is a root?

To ensure the polynomial has rational coefficients, complex roots must come in conjugate pairs. Including -4i guarantees the quadratic factor $(x^2 + 16)$ with rational coefficients.

How can I verify that the polynomial with roots 1, -1, 4i, and -4i has rational coefficients?

By constructing the polynomial as the product of its factors $(x - 1)(x + 1)(x - 4i)(x + 4i)$, and simplifying, you find it equals $x^4 + 15x^2 - 16$, which has rational coefficients. This confirms the polynomial's coefficients are rational.

Additional Resources

What Is A Quartic Polynomial Function With Rational Coefficients And Roots Of 1, -1, And 4i?

Understanding quartic polynomial functions—polynomials of degree four—is a fundamental aspect of algebra that opens doors to advanced mathematical concepts, including roots, coefficients, and polynomial behavior. When specific roots are known, especially those involving complex numbers and rational coefficients, constructing and analyzing such polynomials becomes a fascinating exercise blending algebraic intuition with formal methods. This exploration will delve into the nature, construction, properties, and implications of a quartic polynomial with rational coefficients and roots at 1, -1, and 4i.

Foundation of Polynomial Functions and Roots

What Is a Polynomial Function?

A polynomial function is a mathematical expression involving variables raised to non-negative integer powers, combined with coefficients. The general form of a polynomial of degree n is:

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

where:

- $a_n \neq 0$,
- a_i are coefficients (real or complex),
- n is the degree of the polynomial.

In the context of a quartic polynomial (degree 4), the form is:

$$P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

with $a_4 \neq 0$.

Roots of Polynomial Functions

Roots (or zeros) are values of x where the polynomial evaluates to zero:

$$P(r) = 0$$

- Roots can be real or complex.
- The Fundamental Theorem of Algebra states that a degree n polynomial has exactly n roots in the complex numbers (counting multiplicities).

When roots are specified, they significantly influence the polynomial's structure, especially if coefficients are constrained, such as being rational.

Given Roots: 1, -1, and 4i

The roots provided are:

- $x = 1$
- $x = -1$
- $x = 4i$

Key observations about these roots:

1. Real roots: 1 and -1.
2. Complex root: 4i, which is purely imaginary.
3. Complex conjugate roots: When a polynomial has rational coefficients, complex roots must appear in conjugate pairs. Hence, if 4i is a root, its conjugate, $-4i$, must also be a root to ensure the polynomial's coefficients remain rational.

Implication:
The roots of the polynomial are:

- 1
- -1
- $4i$
- $-4i$

Constructing the Quartic Polynomial with the Given Roots

Step 1: Formulating the Polynomial from Roots

Given that the roots are r_1, r_2, r_3, r_4, the polynomial with these roots (up to a non-zero constant factor) can be expressed as:

$$P(x) = k (x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

where k is a rational (preferably 1 for simplicity) constant that scales the polynomial.

Since the roots are $1, -1, 4i, -4i$, the polynomial becomes:

$$P(x) = k (x - 1)(x + 1)(x - 4i)(x + 4i)$$

Step 2: Simplify the Factors

We can simplify:

- $(x - 1)(x + 1) = x^2 - 1$ (difference of squares).
- $(x - 4i)(x + 4i) = x^2 - (4i)^2$.

Calculating $(4i)^2$:

$$(4i)^2 = 16i^2 = 16 \times (-1) = -16$$

Thus,

$$(x - 4i)(x + 4i) = x^2 - (-16) = x^2 + 16$$

Putting it all together:

$$P(x) = k (x^2 - 1)(x^2 + 16)$$

Step 3: Expand the Polynomial

Expanding:

$$P(x) = k (x^2 - 1)(x^2 + 16)$$

First, perform the multiplication:

$$(x^2 - 1)(x^2 + 16) = x^2 \times x^2 + x^2 \times 16 - 1 \times x^2 - 1 \times 16$$

$$= x^4 + 16x^2 - x^2 - 16$$

$$= x^4 + (16x^2 - x^2) - 16$$

$$= x^4 + 15x^2 - 16$$

Therefore,

$$P(x) = k (x^4 + 15x^2 - 16)$$

Choosing the Constant k and Final Polynomial Form

Coefficient Rationality and Normalization

The polynomial's coefficients are already rational for any rational k. Typically, for simplicity and to have monic polynomials (leading coefficient 1), set $k = 1$:

$$P(x) = x^4 + 15x^2 - 16$$

This polynomial has:

- Rational coefficients,
- Roots at 1, -1, 4i, and -4i,
- Degree four.

Note: If the problem specifies a specific leading coefficient, you can adjust k accordingly.

Verification of Roots

To ensure correctness:

- Check $x=1$:

$$P(1) = 1^4 + 15(1)^2 - 16 = 1 + 15 - 16 = 0$$

- Check $x=-1$:

$$P(-1) = (-1)^4 + 15(-1)^2 - 16 = 1 + 15 - 16 = 0$$

- Check $x=4i$:

$$P(4i) = (4i)^4 + 15(4i)^2 - 16$$

Calculate each term:

- $(4i)^4 = ((4i)^2)^2$

- $(4i)^2 = 16i^2 = 16 \times (-1) = -16$

- $(4i)^4 = (-16)^2 = 256$

Now,

$$P(4i) = 256 + 15 \times (-16) - 16 = 256 - 240 - 16 = 0$$

Similarly, check $x=-4i$:

$$P(-4i) = (-4i)^4 + 15(-4i)^2 - 16$$

- $(-4i)^2 = (-4)^2 \times i^2 = 16 \times (-1) = -16$

- $(-4i)^4 = (-16)^2 = 256$

So,

$$P(-4i) = 256 + 15 \times (-16) - 16 = 256 - 240 - 16 = 0$$

All roots satisfy the polynomial, confirming the correctness.

Properties of the Constructed Quartic Polynomial

Coefficient Rationality

- The polynomial $x^4 + 15x^2 - 16$ has all rational coefficients, satisfying the key condition.
- The coefficients are integers, a subset of rational numbers.

Degree and Roots

- Degree: 4 (quartic).
- Roots: 1, -1, 4i, -4i.
- Roots include both real and complex numbers, with the complex roots forming a conjugate pair, ensuring rational coefficients.

Factorization

The polynomial factors over the complex numbers as:

$$P(x) = (x - 1)(x + 1)(x - 4i)(x + 4i)$$

over the reals, it factors into quadratic polynomials:

$$P(x) = (x^2 - 1)(x^2 + 16)$$

which are both quadratic and have rational coefficients.

Implications and Broader Context

Constructing Polynomials with Prescribed Roots

This process demonstrates a systematic approach:

- Start with known roots.
- Ensure complex roots come in conjugate pairs for rational coefficients.
- Form factors as $(x - r)$.
- Simplify via difference of squares or sum/difference formulas.
- Expand to standard polynomial form.

This method can be generalized for any set of roots, provided the coefficients are constrained to certain domains.

Role of Rational Coefficients in Polynomial Roots

- Rational coefficients impose symmetry in roots—complex roots must come

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