What Is An Equation Of The Tangent Line At X = 2, Assuming That Y(2) = 4 And Y'(2) = 5?

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Understanding the equation of a tangent line at a specific point on a curve is a fundamental concept in calculus. It helps us approximate the behavior of functions locally and provides insights into the function's rate of change at that particular point. In this article, we will explore how to find the equation of the tangent line at $x = 2$, given that $y(2) = 4$ and $y'(2) = 5$. We will walk through the necessary steps, explain key concepts, and provide practical examples to solidify your understanding.

What Is a Tangent Line?

Before diving into calculations, it's essential to understand what a tangent line is and why it matters.

Definition of a Tangent Line

A tangent line to a curve at a specific point is a straight line that touches the curve at that point without crossing it (locally). It represents the best linear approximation of the function near that point, capturing the function's instantaneous rate of change.

Importance of the Tangent Line

- Approximation: Near the point of tangency, the tangent line closely approximates the behavior of the function.
- Understanding Rates: The slope of the tangent line indicates how rapidly the function is increasing or decreasing at that point.
- Applications: Used in optimization, physics (velocity and acceleration), and engineering.

Key Concepts Needed to Find the Equation of the Tangent Line

To find the tangent line at a specific point, you need two critical pieces of information:

1. The Point of Tangency

Given as (x_0, y_0), where in our case,

- $x_0 = 2$
- $y(2) = 4$ (so, the point is $(2, 4)$)

2. The Slope of the Tangent Line

The slope at $x = 2$ is given by the derivative $y'(2)$, which represents the instantaneous rate of change of the function at that point. Here,

- $y'(2) = 5$

Step-by-Step Process to Find the Equation of the Tangent Line

Now that we have the necessary information, let's go through the steps to derive the tangent line equation.

Step 1: Recall the Point and Slope

- Point: $(2, 4)$
- Slope: $m = y'(2) = 5$

Step 2: Use the Point-Slope Form of a Line

The point-slope form is a standard way to write the equation of a line when you know a point and the slope:

$$
y - y_1 = m(x - x_1)
$$

Substituting the known values:

$$
y - 4 = 5(x - 2)
$$

Step 3: Simplify the Equation

Expanding the right side:

\[

$$
y - 4 = 5x - 10
$$

Adding 4 to both sides:

$$
y = 5x - 10 + 4
$$

$$
y = 5x - 6
$$

Final Equation of the Tangent Line

The equation of the tangent line at $x = 2$, given the information, is:

$$
\boxed{ y = 5x - 6 }
$$

This linear equation approximates the behavior of the function near $x = 2$.

Understanding the Components of the Tangent Line Equation

Let's analyze the components of the derived equation:

1. Slope ($m = 5$)

- Indicates the function increases by 5 units in y for every 1 unit increase in x.
- Reflects the instantaneous rate of change at $x = 2$.

2. Y-intercept (-6)

- The point where the tangent line crosses the y-axis.
- Not necessarily on the original curve but part of the linear approximation near $x = 2$.

Visualizing the Tangent Line and the Curve

Graphing the original function and its tangent line offers valuable insight:

Why Visualize?

- To see how well the tangent line approximates the function near $x = 2$.
- To understand the local linearity and behavior of the function.

Plotting Steps

1. Plot the point $(2, 4)$.
2. Draw the tangent line $y = 5x - 6$.
3. Sketch the original curve around $x = 2$ to observe the approximation.

Applications of Finding the Equation of a Tangent Line

Understanding how to compute tangent lines is vital in various fields:

1. Optimization Problems

- Finding maximum or minimum points by analyzing the tangent line’s slope.

2. Physics

- Calculating velocity (instantaneous rate of change) at a specific moment.

3. Engineering and Design

- Modeling how systems behave locally and making predictions.

Additional Examples for Practice

To deepen your understanding, consider applying the process to other points and functions.

Example 1: Quadratic Function

Suppose $y = x^2$, and you want the tangent line at $x = 3$.

- $y(3) = 9$
- $y'(x) = 2x$, so $y'(3) = 6$

Equation:

$$
y - 9 = 6(x - 3) \Rightarrow y = 6x - 18 + 9 = 6x - 9
$$

Example 2: Sine Function

Suppose $y = \sin x$, and at $x = \frac{\pi}{4}$:

- $y = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $y' = \cos x$, so $y' = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Equation:

$$
y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4} \right)
$$

Summary

Finding the equation of a tangent line at a specific point involves understanding the point of tangency and the slope there, which is given by the derivative. Using the point-slope form simplifies the process, leading to a linear approximation of the function near that point. In the context of the initial problem—with $y(2) = 4$ and $y'(2) = 5$—the tangent line is $y = 5x - 6$, providing a powerful tool to analyze and approximate the behavior of functions locally.

Conclusion

Mastering the process of finding tangent lines is essential for calculus students and professionals alike. It forms the foundation for more advanced topics such as derivatives, differentials, and optimization. By understanding the relationship between a function’s value, its derivative, and the tangent line, you gain a deeper insight into the dynamic behavior of mathematical models and real-world systems. Keep practicing with different functions and points to strengthen your skills and intuition in calculus!

Frequently Asked Questions

What is the general formula for the tangent line to a function at a specific point?

The tangent line at a point x = a is given by y = y(a) + y'(a)(x - a), where y(a) is the function value at a and y'(a) is the derivative at a.

How do you find the equation of the tangent line at x = 2 if y(2) = 4 and y'(2) = 5?

Using the point-slope form, the tangent line is y - 4 = 5(x - 2), which simplifies to y = 5x - 6.

What is the slope of the tangent line at x = 2 for this function?

The slope of the tangent line at x = 2 is 5, as given by y'(2) = 5.

What is the y-intercept of the tangent line at x = 2?

Substituting x = 0 into y = 5x - 6 gives y = -6, so the y-intercept is at (0, -6).

How does knowing y(2) and y'(2) help in determining the tangent line?

They provide the point on the function and the slope at that point, enabling the construction of the tangent line's equation.

Can you write the equation of the tangent line in slope-intercept form for this problem?

Yes, the tangent line is y = 5x - 6.

Why is the derivative y'(2) important when finding the tangent line at x = 2?

Because y'(2) gives the slope of the tangent line at x = 2, which is essential for formulating the line's equation.

Additional Resources

Tangent Line Equation: A Deep Dive into Its Derivation and Significance at X = 2

Introduction

In the realm of calculus and analytical geometry, the concept of a tangent line is fundamental. It serves as an invaluable tool for understanding the local behavior of a curve at a specific point. Whether you're a student navigating the basics or an expert applying advanced techniques, grasping how to determine the equation of a tangent line is crucial. This article explores the precise process of deriving the tangent line at a given point—specifically at $x = 2$—when provided with the function's value and derivative at that point.

Imagine you have a function $y = f(x)$, and you're told that at $x = 2$, the function's value is $y(2) = 4$ and its derivative is $y'(2) = 5$. What does this tell us about the behavior of the function near $x = 2$, and how can we formulate the equation of the tangent line? We will walk through every step meticulously, ensuring clarity and depth.

Understanding the Fundamentals: What Is a Tangent Line?

Before diving into calculations, it’s important to understand what a tangent line represents geometrically and analytically.

Geometric Perspective

A tangent line to a curve at a specific point is the straight line that touches the curve exactly at that point, with the same instantaneous direction as the curve’s slope at that point. If you imagine a smooth curve, the tangent line is the best linear approximation of the curve at that point. It "just touches" the curve without crossing it in an infinitesimal neighborhood, providing a local linear approximation of the function.

Analytical Perspective

Mathematically, the tangent line at $x = a$ can be expressed using the point-slope form:

$$
y - y(a) = y'(a)(x - a)
$$

where:

- $y(a)$ is the function value at $x = a$,

- $y'(a)$ is the derivative (slope) at $x = a$.

This formula encapsulates the core idea: the tangent line passes through the point $(a, y(a))$ and has a slope equal to the derivative at that point.

Applying the Concept: Given Data at X = 2

Let’s now contextualize this with the given data:

- $y(2) = 4$

- $y'(2) = 5$

This tells us that at $x = 2$:

- The point on the curve is $(2, 4)$.

- The slope of the curve at that point is 5.

With this information, we are ready to derive the equation of the tangent line.

Step-by-Step Derivation of the Tangent Line Equation

The process involves applying the point-slope form from the fundamentals.

Step 1: Identify the Point of Tangency

The point where the tangent touches the curve is simply $(a, y(a))$:

$$
a = 2, \quad y(a) = 4
$$

Step 2: Determine the Slope at the Point

Given as the derivative:

$$
y'(2) = 5
$$

This indicates the tangent line will have a slope $m = 5$.

Step 3: Write the Equation Using Point-Slope Form

The point-slope form of a line is:

$$
y - y_1 = m (x - x_1)
$$

Plugging in the known values:

$$
y - 4 = 5(x - 2)
$$

Step 4: Simplify to Slope-Intercept Form

Expanding the right side:

$$
y - 4 = 5x - 10
$$

Adding 4 to both sides:

$$
y = 5x - 10 + 4
$$

$$
y = 5x - 6
$$

Final Equation of the Tangent Line:

$$
\boxed{
\textbf{y = 5x - 6}
}
$$

This line touches the curve precisely at $x=2$, passing through the point $(2, 4)$, and has a slope of 5, matching the derivative at that point.

Interpreting the Result: Why Is This Important?

Understanding the tangent line's equation in this context provides several insights:

1. Local Linear Approximation

The tangent line offers a first-order approximation to the function near \(x=2 \). For values of \(x \) close to 2, the function behaves approximately like \(y = 5x - 6 \). This is especially useful in applications like numerical methods, physics, and engineering, where complex functions are approximated locally for analysis.

2. Slope as Instantaneous Rate of Change

The derivative \(y'(2) = 5 \) signifies the instantaneous rate at which \(y \) changes with respect to \(x \) at \(x=2 \). The positive slope indicates an increasing function at that point.

3. Graphical Interpretation

Plotting both the curve \(y = f(x) \) and its tangent line at \(x=2 \) helps visualize how the line approximates the curve locally. The tangent line intersects the curve at exactly one point and shares the same slope, highlighting its role as a linear proxy.

Extending the Concept: Generalization and Practical Applications

The process outlined isn't confined solely to the provided data. It generalizes to any function where the value and derivative at a point are known.

General Formula for the Tangent Line

For a given \(a \), with known:

- \(y(a) \)

- \(y'(a) \)

The tangent line is:

\[
\boxed{
y = y(a) + y'(a)(x - a)
}
\]

This formula is fundamental in differential calculus and forms the basis for numerous approximation techniques, including Taylor series expansions.

Practical Applications

- Physics: Calculating the instantaneous velocity (slope) and position approximation.

- Economics: Estimating marginal cost or revenue at a specific point.

- Engineering: Linearizing complex systems for control and stability analysis.

- Computer Graphics: Rendering curves via tangent approximations.

Potential Challenges and Common Misconceptions

While the derivation appears straightforward, some common pitfalls can arise:

- Confusing the function value and the derivative: Remember, \(y(2) \) is the coordinate on the curve, while \(y'(2) \) is the slope.

- Applying the wrong point: Always use the point corresponding to the specified \(x \)-value.

- Misinterpreting the tangent as a secant: The tangent line only touches the curve at one point and has the same slope there; it does not cross the curve in an infinitesimal neighborhood.

- Neglecting the domain considerations: The tangent line is a local approximation; it may not represent the behavior of the function far from \(x=2 \).

Conclusion

The equation of the tangent line at \(x=2 \) for a function with known point and slope data encapsulates the essence of calculus: understanding how a function behaves locally through its derivatives. Here, with \(y(2) = 4 \) and \(y'(2) = 5 \), the tangent line is succinctly expressed as:

\[

\boxed{
\text{Tangent Line: } y = 5x - 6
}
\]

This line not only provides a linear approximation of the function near \(x=2 \) but also exemplifies the powerful connection between derivatives and tangent lines. Mastery of this concept underpins many advanced topics across mathematics, physics, engineering, and beyond, serving as a cornerstone for analytical reasoning and practical problem-solving.

In essence, understanding and deriving the tangent line at a specific point is a fundamental skill that bridges the gap between calculus theory and real-world applications, offering profound insights into the behavior of functions at precise moments.

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