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Understanding differential equations is fundamental in advanced mathematics, especially when modeling real-world phenomena such as physics, engineering, and economics. Among these, homogeneous differential equations hold a special place due to their unique structure and solution techniques. This article provides a comprehensive guide on rewriting the differential equation $(5x^2 - 2y^2)\,dx = xy\,dy$ in the form $dy/dx = f(y/x)$, illustrating the process step-by-step and emphasizing the importance of homogeneous equations in mathematical analysis.

Introduction to Homogeneous Differential Equations

Homogeneous differential equations are a class of differential equations characterized by their specific form, which allows them to be simplified using substitution methods. These equations typically involve functions where the degrees of the numerator and denominator are the same, enabling the substitution $y = vx$ or $x = vy$ to reduce the equation to a separable form.

Key features of homogeneous differential equations include:

- Both the differential equation's numerator and denominator are homogeneous functions of the same degree.
- They often can be written in the form $dy/dx = f(y/x)$ or $dx/dy = g(x/y)$.
- They are solvable using substitution methods, notably $y = vx$ or $x = vy$.

Understanding the Given Differential Equation

The differential equation under consideration is:

$$(5x^2 - 2y^2)\,dx = xy\,dy$$

The equation involves quadratic terms of both x and y, suggesting that it may be homogeneous. Our goal is to express it in the form $dy/dx = f(y/x)$, which simplifies the process of solving the differential equation.

Steps to analyze:

1. Identify the structure: Recognize whether the given form is homogeneous
2. Rearrange the equation: Express dy/dx explicitly
3. Apply substitution: Use $y = vx$ or $x = vy$ to reduce the equation

Rearranging the Differential Equation

The initial step involves rewriting the given differential equation to isolate dy/dx:

$$(5x^2 - 2y^2)\, dx = xy\, dy$$

Divide both sides of the equation by dx:

$$(5x^2 - 2y^2) = xy\, (dy/dx)$$

Hence, the derivative dy/dx is:

$$dy/dx = (5x^2 - 2y^2) / (xy)$$

This form is close to the desired format, but it involves both x and y explicitly. To express dy/dx as a function of y/x, we need to manipulate the numerator to factor out x^2 and y^2 terms.

Expressing dy/dx in Terms of y/x

The goal is to rewrite $dy/dx = f(y/x)$. To do this, introduce a substitution:

Let $v = y / x$

which implies:

$$y = v\, x$$

Differentiating both sides with respect to x:

$$dy/dx = v + x\, dv/dx$$

Now, substitute $y = v\, x$ into the expression for dy/dx:

$$dy/dx = (5x^2 - 2(v\, x)^2) / (x\, v\, x) = (5x^2 - 2v^2\, x^2) / (x^2\, v)$$

Simplify numerator and denominator:

$dy/dx = [x^2 (5 - 2v^2)] / (x^2 v) = (5 - 2v^2) / v$

Recall that $dy/dx = v + x\, dv/dx$, thus:

$v + x\, dv/dx = (5 - 2v^2) / v$

Rearranged to solve for $x\, dv/dx$:

$x\, dv/dx = (5 - 2v^2) / v - v = [(5 - 2v^2) - v^2] / v = (5 - 3v^2) / v$

This expression involves only v and dv/dx, which simplifies the problem to a separable differential equation in v and x.

Final form:

$x\, dv/dx = (5 - 3v^2) / v$

or equivalently:

$dv/dx = (5 - 3v^2) / (v x)$

This form indicates that the original differential equation can be transformed into an equation involving the ratio y/x, confirming that it is homogeneous.

Expressing the Differential Equation as Dy/Dx = f(y/x)

Recall that:

$dy/dx = v + x\, dv/dx$

From earlier, we have:

$v + x\, dv/dx = (5 - 2v^2) / v$

Substitute $x\, dv/dx$ from the previous step:

$v + (5 - 3v^2) / v = dy/dx$

Simplify the left side:

$v + (5 - 3v^2) / v = (v^2 + 5 - 3v^2) / v = (5 - 2v^2) / v$

Thus, the derivative dy/dx in terms of v (which is y/x) is:

$dy/dx = (5 - 2v^2) / v$

Since v = y / x, we can write:

$dy/dx = f(y/x) = (5 - 2 (y/x)^2) / (y/x)$

which simplifies to:

$dy/dx = (5 - 2 (y/x)^2) (x / y)$

This shows that dy/dx is expressed explicitly as a function of y/x, fulfilling the goal.

Summary:

- The original differential equation is homogeneous.
- By substitution y = vx, we reduce it to a separable form.
- The resulting form is dy/dx = f(y/x), where:

$f(y/x) = (5 - 2(y/x)^2) / (y/x)$

or equivalently,

$dy/dx = (5 - 2 (y/x)^2) (x / y)$

Significance of Homogeneous Differential Equations

Homogeneous equations like the one discussed are significant because they:

- Enable substitution methods that simplify complex equations
- Lead to solutions involving functions of y/x, which often have geometric interpretations
- Are common in physics, such as in problems involving proportional relationships or scale invariance

Advantages include:

- Reduced complexity in solving differential equations
- Easier integration once expressed in the correct form
- Clearer understanding of the relationship between variables

Methodology for Solving Homogeneous Differential Equations

Once the differential equation is expressed as $dy/dx = f(y/x)$, solving proceeds through the following steps:

1. Substitute $y = vx$, leading to $dy/dx = v + x\, dv/dx$
2. Rewrite the differential equation in terms of v and x
3. Separate variables if possible, leading to an integral in v and x
4. Integrate both sides to find v as a function of x
5. Back-substitute $y = vx$ to find y in terms of x

This systematic approach simplifies the process of solving such equations and enhances understanding of the relationships between variables.

Practical Applications of the Differential Equation

The ability to express and solve homogeneous differential equations like $(5x^2 - 2y^2)\, dx = xy\, dy$ has numerous real-world implications:

- Physics: Modeling phenomena where quantities scale proportionally, such as in thermodynamics or mechanics.
- Economics: Analyzing proportional relationships between variables like supply and demand.
- Engineering: Designing systems where variables change proportionally, such as in control systems.

These applications highlight the importance of mastering the transformation of differential equations into the form $dy/dx = f(y/x)$.

Conclusion

Transforming the differential equation $(5x^2 - 2y^2)\, dx = xy\, dy$ into the form $dy/dx = f(y/x)$ involves recognizing its homogeneous nature, applying the substitution $y = vx$, and simplifying to express the derivative explicitly as a function of y/x. This process not only simplifies solving the differential equation but also deepens understanding of the relationships between variables in scaled systems. Homogeneous differential equations are fundamental in various fields, offering a powerful tool for modeling and analysis. Mastery of these techniques enhances problem-solving skills and provides insights into the behavior of complex systems modeled by differential equations.

Summary of Key Steps:

1. Recognize the equation's homogeneous nature.
2. Rewrite to isolate dy/dx.
3. Substitute y = vx and differentiate.
4. Simplify to express dy/dx as a function of y/x.
5. Solve the resulting separable differential equation.

By following these steps, students and professionals can efficiently analyze and solve a wide range of homogeneous differential equations, advancing their mathematical proficiency and application skills.

Frequently Asked Questions

How can I rewrite the differential equation (5x^2 - 2y^2) dx = xy dy in the form dy/dx = f(y/x)?

To rewrite the equation in the form dy/dx = f(y/x), divide both sides by dx and rearrange: dy/dx = [(5x^2 - 2y^2) / (xy)]. Then, introduce a substitution u = y/x, so y = ux, and express everything in terms of u and x to obtain dy/dx as a function of y/x.

What substitution is typically used to convert the given differential equation into a homogeneous form?

The substitution u = y/x is used to convert the differential equation into a homogeneous form because it simplifies the variables, allowing the equation to be expressed solely in terms of u and x, leading to dy/dx = f(u).

How do I verify if a differential equation is homogeneous and suitable for substitution y/x?

A differential equation is homogeneous if all terms can be expressed as functions of y/x or if dividing the entire equation by a common factor yields a form where each term is of the same degree. If the equation can be written with all terms involving variables in the form of y/x, it is suitable for substitution u = y/x.

Can you provide the step-by-step process to transform $(5x^2 - 2y^2)$ dx = xy dy into dy/dx = f(y/x)?

Yes. First, rewrite the equation as $dy/dx = [(5x^2 - 2y^2) / (xy)]$. Next, substitute $y = ux$, so $y/x = u$, and $y^2 = u^2 x^2$. Then, express everything in terms of u and x: $dy/dx = [(5x^2 - 2u^2 x^2) / (x u x)] = [x^2 (5 - 2u^2)] / (x^2 u)$. Simplify to get $dy/dx = (5 - 2u^2) / u$, which is a function of $u = y/x$.

What are the benefits of rewriting a differential equation in the form dy/dx = f(y/x)?

Rewriting the differential equation in the form $dy/dx = f(y/x)$ simplifies the problem by enabling substitution $u = y/x$, transforming it into a separable differential equation in terms of u and x. This approach often makes solving the equation more straightforward.

Additional Resources

Homogeneous Differential Equation: An In-Depth Exploration of the Equation $(5x^2 - 2y^2)$ dx = xy dy and Its Transformation into the Form dy/dx = f(y/x)

Introduction

Differential equations are foundational tools in mathematics, physics, engineering, and numerous applied sciences. They describe relationships involving functions and their derivatives, often modeling real-world phenomena such as motion, heat transfer, and population dynamics. Among the various types, homogeneous differential equations occupy a special place due to their structure and solvability techniques. These equations exhibit a particular symmetry that allows for substitution strategies simplifying their solution process.

In this article, we delve into a specific homogeneous differential equation:

$$
(5x^2 - 2y^2) \, dx = xy \, dy
$$

Our goal is to analyze this equation thoroughly, demonstrate its homogeneous nature, transform it into the canonical form:

$$
\frac{dy}{dx} = f\left(\frac{y}{x}\right)
$$

\]

and explore the implications, solution methods, and applications associated with such equations. The discussion will be systematic, covering definitions, mathematical transformations, solution techniques, and contextual significance, all structured to enrich understanding and foster analytical skills.

Understanding Homogeneous Differential Equations

What Is a Homogeneous Differential Equation?

A differential equation is termed homogeneous if it exhibits a specific scaling property. For first-order equations, this generally means that the functions involved are homogeneous functions of the same degree. More precisely, a differential equation of the form:

\[
dy/dx = F(x, y)
\]

is homogeneous if the function \(F(x, y)\) can be expressed solely in terms of the ratio \(y/x\). Mathematically, this entails:

\[
F(tx, ty) = F(x, y)
\]

for all non-zero \(t\). This property hints at the equation's invariance under scaling transformations, a feature that simplifies solving such equations through substitution.

Significance of Homogeneity

The homogeneity property allows for the substitution:

\[
v = \frac{y}{x}
\]

which transforms the original differential equation into an equation in terms of \(v\) and \(x\), often reducing it to a separable form. This method leverages the symmetry of the equation, making complex relationships more manageable.

Analyzing the Given Differential Equation

Presentation of the Equation

The equation under consideration is:

$$
(5x^2 - 2y^2) \, dx = xy \, dy
$$

which can be rewritten as:

$$
(5x^2 - 2y^2) \, dx - xy \, dy = 0
$$

This form is conducive to identifying its homogeneous nature and transforming it into a more solvable structure.

Checking for Homogeneity

To verify whether this differential equation is homogeneous, we analyze the functions involved:

$$
M(x, y) = 5x^2 - 2y^2, \quad N(x, y) = -xy
$$

A function $F(x, y)$ is homogeneous of degree n if:

$$
F(tx, ty) = t^n F(x, y)
$$

Let's check M and N:

- For $M(tx, ty)$:

$$
5(tx)^2 - 2(ty)^2 = 5t^2 x^2 - 2t^2 y^2 = t^2 (5x^2 - 2 y^2)
$$

- For $N(tx, ty)$:

$$
- (tx)(ty) = - t^2 xy
$$

Both M and N are homogeneous functions of degree 2, confirming the entire differential equation's homogeneity.

Transforming the Equation into $\boldsymbol{\frac{dy}{dx} = f(y/x)}$

Motivation for the Substitution $v = y/x$

The homogeneity of the functions suggests using the substitution:

$$
v = \frac{y}{x}
$$

which simplifies the dependence of the equation on x and y, enabling us to express derivatives in terms of v and x.

Deriving $\frac{dy}{dx}$ in Terms of v

Given $v = y/x$, then:

$$
y = v x
$$

Differentiating both sides with respect to x:

$$
\frac{dy}{dx} = v + x \frac{dv}{dx}
$$

This relation allows rewriting the original differential equation in terms of v and x.

Step-by-Step Transformation Process

Rewrite $M(x, y)$ and $N(x, y)$ in terms of v

- Since $y = v x$:

$$
M(x, y) = 5x^2 - 2 (v x)^2 = 5x^2 - 2 v^2 x^2 = x^2 (5 - 2 v^2)
$$

- Similarly,

$$
N(x, y) = - x y = - x (v x) = - v x^2
$$

The differential equation becomes:

$$
[x^2 (5 - 2 v^2)] dx - v x^2 dy = 0
$$

Dividing through by x^2:

$$
(5 - 2 v^2) dx - v dy = 0
$$

Recall that:

$$
dy/dx = v + x dv/dx
$$

which can be substituted into the rewritten equation:

$$
(5 - 2 v^2) dx - v (dy) = 0
$$

Expressed as:

$$
(5 - 2 v^2) dx - v (v + x dv/dx) dx = 0
$$

since $dy = (v + x dv/dx) dx$.

Simplify:

$$
(5 - 2 v^2) dx - v (v + x dv/dx) dx = 0
$$

$$
\left[(5 - 2 v^2) - v^2 - v x \frac{dv}{dx} \right] dx = 0
$$

which simplifies to:

$$
(5 - 3 v^2) - v x \frac{dv}{dx} = 0
$$

Rearranged as:

$$
v x \frac{dv}{dx} = 5 - 3 v^2
$$

or

$$
\frac{dv}{dx} = \frac{5 - 3 v^2}{v x}
$$

Final Form: $\mathbf{\frac{dy}{dx} = f(y/x)}$

Recall that:

$$
\frac{dy}{dx} = v + x \frac{dv}{dx}
$$

Substitute $\frac{dv}{dx}$:

$$
\frac{dy}{dx} = v + x \times \frac{5 - 3 v^2}{v x} = v + \frac{5 - 3 v^2}{v}
$$

Simplify:

\[
\frac{dy}{dx} = v + \frac{5 - 3 v^2}{v} = \frac{v^2 + 5 - 3 v^2}{v} = \frac{5 - 2 v^2}{v}
\]

Expressed in terms of \(v = y/x\):

\[
\boxed{
\frac{dy}{dx} = \frac{5 - 2 \left(\frac{y}{x}\right)^2}{\frac{y}{x}} = \frac{5 - 2 (y/x)^2}{(y/x)}
}
\]

which simplifies to:

\[
\boxed{
\frac{dy}{dx} = \frac{5}{(y/x)} - 2 \frac{(y/x)^2}{(y/x)} = \frac{5}{(y/x)} - 2 (y/x)
}
\]

or more neatly:

\[
\boxed{
\frac{dy}{dx} = \frac{5}{v} - 2 v
}
\]

This is the canonical form:

\[
\frac{dy}{dx} = f\left(\frac{y}{x}\right) = \frac{5}{v} - 2 v
\]

which explicitly shows dependence only on \(v = y/x\).

Significance of the Transformation

Transforming the original differential equation into the form:

\[
\frac{dy}{dx} = f\left(\frac{y}{x}\right)
\]

has multiple advantages:

- Reduction to a separable equation: Since \(f(v)\) depends solely on \(v\), the differential equation becomes easier to solve by substitution.
- Insight into scaling symmetry: The form indicates invariance under scaling transformations, characteristic of homogeneous equations.
- Simplification of the solution process: The substitution \(v = y/x\) reduces the problem to a first-order ordinary differential equation in a single variable.

Solving the Transformed Equation

Step 1: Write the differential equation in terms of \(v\)

From earlier,

\[
\frac{dy}{dx} =

Write The Homogeneous Differential Equation 5x 2 2y 2 Dx Xydy In The Form Dy Dx F Y X

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