15. If F:G+ G Is A Homomorphism Of Groups, Then Prove That F = {a E Gf(a) = A} Is A Subgroup Of G

15. If F:G+ G Is A Homomorphism Of Groups, Then Prove That F = {a E Gf(a) = A} Is A Subgroup Of G

Understanding the structure of groups and group homomorphisms is fundamental in abstract algebra. The statement under consideration explores a crucial property: given a group homomorphism $f: G \to G$, the set $F = \{a \in G \mid f(a) = A\}$ forms a subgroup of G. In this article, we will delve deep into the proof of this statement, exploring the necessary concepts, definitions, and logical steps involved.

Preliminaries and Definitions

Before diving into the proof, it's essential to understand some core concepts related to groups and homomorphisms.

What is a Group?

A group is a set G equipped with a binary operation $\cdot$ satisfying the following properties:

- Closure: For all $a, b \in G$, $a \cdot b \in G$.
- Associativity: For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Identity element: There exists an element $e \in G$ such that for all $a \in G$, $e \cdot a = a \cdot e = a$.
- Inverse element: For each $a \in G$, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

What is a Group Homomorphism?

A homomorphism between two groups G and H is a function $f: G \to H$ satisfying:

$$
f(a \cdot b) = f(a) \cdot' f(b) \quad \text{for all } a, b \in G,
$$

where $\cdot$ and $\cdot'$ are the group operations in G and H, respectively. In the context of the problem, the homomorphism is from a group G to itself, i.e., $f: G \to G$.

Key Properties of Group Homomorphisms

- Preservation of identity: $f(e_G) = e_G$ or possibly other elements if f is not an isomorphism.
- Preservation of inverses: $f(a^{-1}) = (f(a))^{-1}$.
- Kernel of f: $\ker f = \{a \in G \mid f(a) = e\}$, which is always a subgroup of G.

In this context, the notation $f(a) = A$ indicates that the homomorphism maps a to a specific element A in G.

Understanding the Set $F = \{a \in G \mid f(a) = A\}$

Given a homomorphism $f: G \to G$, and a fixed element $A \in G$, the set

$$
F = \{a \in G \mid f(a) = A\}
$$

is called a fiber or preimage of A under f. The main goal is to demonstrate that F is a subgroup of G.

Proving that F is a Subgroup of G

To establish that F is a subgroup, we need to verify the subgroup criteria:

1. Non-empty: $F \neq \emptyset$
2. Closure under the group operation: For any $a, b \in F$, $a \cdot b \in F$
3. Closure under inverses: For any $a \in F$, $a^{-1} \in F$

Let's analyze these in detail.

Step 1: Show that F is Non-Empty

Since f is a homomorphism, it maps the identity element $e \in G$ to some element $f(e)$.

- Observation: For any homomorphism $f: G \to G$,

$$
f(e) = e,

\]

if f is a group homomorphism from G to itself (which is often the case).

- Implication: If $A = f(e)$, then $e \in F$, because

$$
f(e) = A,
$$

making the set F non-empty.

- General case: If $A \neq f(e)$, then F could be empty unless A is in the image of f. For the purposes of the proof, we consider that $A \in \operatorname{Im}(f)$, ensuring F is non-empty.

Conclusion: There exists at least one element $a \in G$ such that $f(a) = A$, so $F \neq \emptyset$.

Step 2: Closure Under the Group Operation

Suppose $a, b \in F$. Then:

$$
f(a) = A, \quad f(b) = A.
$$

We need to verify whether $a \cdot b \in F$, i.e., whether $f(a \cdot b) = A$.

- Using the homomorphism property:

$$
f(a \cdot b) = f(a) \cdot f(b) = A \cdot A.
$$

- Analysis:

$$
f(a \cdot b) = A \cdot A,
$$

which equals A if and only if $A \cdot A = A$.

Implication:

- For F to be closed under the group operation, the element A must satisfy:

$$
A \cdot A = A,

\]

which implies A is an idempotent element under the group operation.

- Note: In a group, the only idempotent element is the identity e, because:

$$
A \cdot A = A \Rightarrow A \cdot A = A \Rightarrow A \cdot A = A \Rightarrow A \cdot A \cdot A^{-1} = A \cdot A^{-1} \Rightarrow A = e,
$$

since multiplying both sides on the right by A^{-1} yields:

$$
A \cdot A = A \Rightarrow A \cdot A \cdot A^{-1} = A \cdot A^{-1} \Rightarrow A = e.
$$

Therefore:

- If $A = e$, then for any $a, b \in F$,

$$
f(a) = e, \quad f(b) = e,
$$

and

$$
f(a \cdot b) = f(a) \cdot f(b) = e \cdot e = e,
$$

so $a \cdot b \in F$.

- If $A \neq e$, the set F contains elements mapped to A, but the closure may not hold unless A is the identity.

Conclusion:

- The set F is closed under the group operation if and only if $A = e$.

Step 3: Closure Under Inverses

Suppose $a \in F$, so $f(a) = A$. To show $a^{-1} \in F$, we examine:

$$
f(a^{-1}) = (f(a))^{-1} = A^{-1}.
$$

- Implication:

$$
a^{-1} \in F \quad \text{if and only if} \quad f(a^{-1}) = A,
$$

which is equivalent to:

$$
A^{-1} = A.
$$

- Therefore, for $a^{-1} \in F$, the element A must satisfy:

$$
A^{-1} = A,
$$

meaning A is its own inverse.

- In a group, elements satisfying $A = A^{-1}$ are elements of order 1 or 2, i.e., elements of order dividing 2.

Summary of Conditions for F to be a Subgroup

From the above analysis, the set

$$
F = \{a \in G \mid f(a) = A\}
$$

is a subgroup of G if and only if:

1. $A = e$, the identity element, ensuring non-emptiness and closure under the group operation.
2. $A = A^{-1}$, which is automatically true if $A = e$.

Frequently Asked Questions

What is the definition of a homomorphism between groups F and G?

A homomorphism between groups F and G is a function f: F → G such that for all a, b in F, f(ab) = f(a)f(b).

How is the set $F = \{a \in G \mid f(a) = a\}$ defined in the context of a homomorphism $f: G \to G$?

F is the set of all elements a in G for which the homomorphism f maps a to itself, i.e., $f(a) = a$.

What needs to be proven to show that F is a subgroup of G?

We need to verify that F is non-empty, closed under the group operation, and closed under taking inverses.

Why is the identity element e of G always in the set F?

Because $f(e) = e$ for any homomorphism f (since homomorphisms map identity to identity), so $e \in F$.

How do we show that F is closed under the group operation?

For any $a, b \in F$, since $f(a) = a$ and $f(b) = b$, then $f(ab) = f(a)f(b) = ab$, so $ab \in F$.

How do we verify that F is closed under inverses?

For any $a \in F$, since $f(a) = a$, then $f(a^{-1}) = f(a)^{-1} = a^{-1}$, so $a^{-1} \in F$.

What is the significance of the set F in the context of group homomorphisms?

F is the fixed point set of the homomorphism, often called the kernel when f maps to the identity, and it forms a subgroup of G.

Can F be the entire group G? Under what condition?

Yes, if f is the identity homomorphism, then F = G, since every element maps to itself.

Is F always a normal subgroup of G? Why or why not?

Not necessarily; F is a subgroup, but it is normal if f is a group homomorphism that is also a normal map, such as a kernel of a homomorphism. In this context, F as defined may not always be normal unless specified.

What conclusion can we draw about F given that $f: G \to G$ is a homomorphism with $f(a) = a$ for all a in F?

F is a subgroup of G consisting of elements fixed by the homomorphism f, and it can be shown to satisfy subgroup properties, thus confirming that F is indeed a subgroup.

Additional Resources

Homomorphism and Subgroup Formation: An In-Depth Analysis of the Set F = {a ∈ G | f(a) = a}

Introduction to Homomorphisms and Subgroups

Mathematics, particularly abstract algebra, explores structures like groups, rings, and fields. Among these, groups serve as fundamental building blocks for understanding symmetry, algebraic operations, and various transformations. A core concept within group theory is the notion of homomorphisms, which are structure-preserving maps between groups.

Understanding how these mappings relate to the substructures within groups—particularly subgroups—is central to grasping the deeper properties of algebraic systems. In this discussion, we analyze the set $F = \{ a \in G \mid f(a) = a \}$, where $f: G \to G$ is a homomorphism, and demonstrate that this set forms a subgroup of G.

Fundamentals of Group Homomorphisms

Definition of a Group Homomorphism

A group homomorphism is a function $f: G \to H$ between two groups G and H such that for all elements $a, b \in G$:

$$
f(ab) = f(a)f(b)
$$

This property ensures that the operation structure of the domain group G is preserved under f. Homomorphisms can be viewed as structure-preserving maps, mapping elements and operations from one group to another in a consistent manner.

Key Properties of Homomorphisms

- Identity Preservation: For a homomorphism f,

$$
f(e_G) = e_H
$$

where e_G and e_H are the identity elements of G and H, respectively.

- Kernel of a Homomorphism: The set

$$
\ker(f) = \{ a \in G \mid f(a) = e_H \}
$$

is a normal subgroup of G. It captures elements mapped to the identity of the codomain.

- Image of a Homomorphism: The set

$$
\operatorname{Im}(f) = \{ f(a) \mid a \in G \}
$$

is a subgroup of H.

- Homomorphism Preservation of Inverses: For any $a \in G$,

$$
f(a^{-1}) = [f(a)]^{-1}
$$

which follows from the homomorphism property.

The Set $F = \{a \in G \mid f(a) = a\}$: An Intuitive Overview

Given a homomorphism $f: G \to G$, the set

$$
F = \{ a \in G \mid f(a) = a \}
$$

is known as the set of fixed points of f. These are elements of G that remain unchanged under the map f.

This set is of particular interest because it captures the elements that are invariant under

f. The question at hand is whether this set forms a subgroup of G.

The intuition suggests that, because f preserves the group structure, the set of elements fixed by f should be closed under group operation and inverses, and include the identity, thus satisfying the subgroup criteria.

Proving F is a Subgroup of G

To establish that F is a subgroup of G, we will verify the standard subgroup criteria:

1. The identity element e of G is in F.
2. F is closed under the group operation.
3. F is closed under inverses.

Let’s analyze each step carefully.

Step 1: Identity Element in F

- Since f is a homomorphism, it preserves the identity:

$$
f(e) = e
$$

- Therefore,

$$
f(e) = e \implies e \in F
$$

- Conclusion: The identity element e of G is in F.

Step 2: Closure Under Group Operation

We need to show that for any $a, b \in F$:

$$
ab \in F
$$

which means:

$$
f(ab) = ab
$$

- Since $a, b \in F$, by definition:

$$
f(a) = a \quad \text{and} \quad f(b) = b
$$

- Using the homomorphism property:

$$
f(ab) = f(a)f(b) = a b
$$

- Therefore,

$$
f(ab) = a b
$$

which implies $ab \in F$.

- Conclusion: F is closed under the group operation.

Step 3: Closure Under Inverses

We need to verify that for any $a \in F$:

$$
a^{-1} \in F
$$

which requires:

$$
f(a^{-1}) = a^{-1}
$$

- Since $a \in F$, $f(a) = a$.

- Applying the homomorphism property to a^{-1}:

$$
f(a^{-1}) = [f(a)]^{-1} = a^{-1}
$$

- Conclusion: $a^{-1} \in F$.

Summary of the Proof

Combining all three steps, we find that:

- The identity element e belongs to F.
- F is closed under the group operation.
- F is closed under taking inverses.

By the subgroup criterion, F is a subgroup of G.

Additional Insights and Implications

Normality of F

While F is a subgroup, it is not necessarily a normal subgroup of G. Normality depends on whether $gFg^{-1} = F$ for all $g \in G$. Generally, the set of fixed points under a homomorphism need not be normal unless additional conditions are met.

Relation to the Kernel of f

- The kernel $\ker(f) = \{ a \in G \mid f(a) = e \}$ is always a normal subgroup.

- The set F is a subset of G where $f(a) = a$, which may or may not include the kernel.

- For the identity element, e, we have:

$$
f(e) = e
$$

so $e \in F$.

- The kernel contains elements mapped to e, which is generally not the same as the fixed point set unless f is the identity map.

Special Cases

- If f is the identity map id_G, then:

$$
F = \{ a \in G \mid a = f(a) = a \} = G
$$

and trivially, F is a subgroup (the entire group).

- If f is the trivial homomorphism $f(a) = e$ for all a, then:

$$
F = \{ a \in G \mid e = a \} = \{ e \}
$$

which is a subgroup (the trivial subgroup).

- For more general homomorphisms, the fixed point set F captures elements invariant under f, serving as an important algebraic structure in understanding automorphisms and endomorphisms.

Broader Context and Applications

Automorphisms and Fixed Points

- When f is an automorphism (a bijective homomorphism from G to itself), the set F of fixed points is crucial in symmetry analysis.

- Fixed point subgroups can help classify automorphisms, especially in finite groups, and analyze group actions.

Endomorphisms and Invariant Subgroups

- The set F is an example of an invariant subgroup under f.

- The concept extends to various algebraic structures, where invariance under particular maps leads to substructure analysis.

Relation to Group Actions

- The fixed point set can also be viewed in the context of group actions, where elements fixed under a group automorphism correspond to elements invariant under the action.

Conclusion

In conclusion, the set $F = \{ a \in G \mid f(a) = a \}$, where $f: G \to G$ is a homomorphism, is a subgroup of G. The proof hinges on verifying the classical subgroup criteria—identity inclusion, closure under operation, and closure under inverses—each of which follows naturally from the

15 If F G G Is A Homomorphism Of Groups Then Prove That F A E Gf A A Is A Subgroup Of G

15 If F G G Is A Homomorphism Of Groups Then Prove That F A E Gf A A Is A Subgroup Of G

Related Articles

- Isotopes Of A Particular Element Differ In The Number Of ___________, And Therefore Differ In Mass
- A Fourth Of All Adopted Children In The United States Were Born In Other Countries. True Or False
- What Is The Solution To The Equation? Square Root -2x-5-4=x A. -7 And -3 B. 3 And 7 C. -3 D. 7

Back to Home