2) Uxy Da, Where D Is The Region In The First Quadrant Bounded By The Parabolas X = Y And X = 8 Y?

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This problem involves evaluating a double integral over a specific region in the first quadrant defined by two parabolas: $X = Y$ and $X = 8Y$. The goal is to understand the nature of the region D, set up the integral appropriately, and compute its value. To do this effectively, a firm grasp of the geometric boundaries, coordinate transformations, and integration techniques is essential.

Understanding the Region D

Defining the Boundaries

The region D is bounded by the two parabolas:

- $X = Y$, which is a straight line passing through the origin with slope 1.
- $X = 8Y$, another straight line passing through the origin with a steeper slope of 8.

Both curves lie in the first quadrant, where $X > 0$ and $Y > 0$. The boundaries intersect at the origin $(0,0)$, and since both are lines through the origin, the region D is the wedge between these two lines extending into the first quadrant.

Visualizing the Region

Plotting the two lines:

- $X = Y$ is a line at 45°, dividing the first quadrant into two regions.

- $X = 8Y$ is steeper, passing through the origin with a slope of 8.

> The region D is the area between these two lines, for $Y > 0$, where Y ranges from 0 outward, and X varies between these two lines at each fixed Y.

Describing the Region Mathematically

For a fixed Y, the bounds on X are:

$$
X = Y \quad \text{(lower boundary)} \\
X = 8Y \quad \text{(upper boundary)}
$$

Since both bounds are functions of Y, and the region extends infinitely into the first quadrant, the limits for Y are from 0 to infinity, but in practical calculations, we consider a finite region or convergence conditions depending on the integrand.

Choosing an Appropriate Coordinate System

Why Consider Change of Variables?

The boundaries are given by linear functions of Y, but they are expressed in terms of X and Y. Transforming to a different coordinate system can simplify the integral by turning the boundaries into simple lines or rectangles.

Transforming to New Variables

Let’s define the transformation:

$$
u = X / Y \quad \text{or} \quad v = Y
$$

or consider a more suitable substitution to linearize the boundaries. Since the boundaries are $X = Y$ and $X = 8Y$, dividing X by Y to get a ratio is a natural choice.

Proposed Substitution

- Set $u = X / Y$, which simplifies the boundaries:
  - $X = Y$ becomes $u = 1$

    - $X = 8Y$ becomes $u = 8$

- Set $v = Y$, preserving the Y coordinate.

Thus, the transformation:

$$
X = u v, \quad Y = v
$$

with u ranging from 1 to 8, and $v > 0$. The region D maps to the rectangle in the u-v plane with:

$$
u \in [1, 8], \quad v \in (0, \infty)
$$

Computing the Jacobian of the Transformation

Jacobian Calculation

From the transformation:

$$
X = u v, \quad Y = v
$$

we compute the Jacobian determinant:

$$
J = \left| \frac{\partial (X, Y)}{\partial (u, v)} \right|
$$

which is:

$$
J = \begin{vmatrix}
\frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\
\frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v}
\end{vmatrix}
= \begin{vmatrix}
v & u \\
0 & 1
\end{vmatrix}
= v \times 1 - 0 \times u = v
$$

Therefore, the differential area element transforms as:

$$
dA = |J| du dv = v du dv
$$

Setting Up the Integral

Original Integral Expression

Suppose the integrand is a function $f(X, Y)$. The double integral over D is:

$$
\iint_D f(X, Y) dA
$$

Rewritten in the $u-v$ Coordinates

Using the substitution:

$$
X = u v, \quad Y = v, \quad dA = v du dv
$$

the integral becomes:

$$
\int_{v=0}^{\infty} \int_{u=1}^{8} f(u v, v) \times v \, du \, dv
$$

Choosing an Integrand

For illustration, consider a specific integrand, for example, $f(X, Y) = 1$, which evaluates the area of the region D. The integral simplifies to:

$$
\iint_D dA = \int_{v=0}^{\infty} \int_{u=1}^{8} v \, du \, dv
$$

Evaluating the Integral

Area Calculation for $f(X, Y) = 1$

The integral reduces to:

$$
\int_{v=0}^{\infty} \left(\int_{u=1}^{8} v \, du \right) dv
= \int_{v=0}^{\infty} v \left[u \right]_{u=1}^{8} dv
= \int_{v=0}^{\infty} v (8 - 1) dv
= 7 \int_{0}^{\infty} v \, dv
$$

which diverges to infinity, indicating the region is unbounded and the area is infinite.

Considering Convergence with a General Integrand

If the integrand involves exponential decay or other damping factors, the integral may converge. For example, suppose:

$$
f(X, Y) = e^{-\alpha X - \beta Y}
$$

then the integral becomes:

$$
\int_{v=0}^{\infty} \int_{u=1}^{8} e^{-\alpha u v - \beta v} v \, du \, dv
$$

which can be integrated step-wise to find a finite value, depending on the parameters $\alpha, \beta > 0$.

Summary and Conclusions

Key Points

- The region D in the first quadrant bounded by $X = Y$ and $X = 8Y$ is a wedge extending to infinity, with unbounded area unless the integrand ensures convergence.

- Transforming to the variables $u = X / Y$ and $v = Y$ simplifies the boundaries to a rectangular region $1 \leq u \leq 8$, $v \geq 0$.

- The Jacobian determinant for this transformation is $|J| = v$, which must be included when changing variables in the integral.

- Depending on the integrand, the double integral can be evaluated explicitly or may diverge, indicating an unbounded region.

Further Considerations

1. For finite integrals, consider restricting the region D to a bounded subset, such as v from 0 to some finite V.

2. Explore other substitutions if the integrand or region shape suggests alternative approaches.

3. Use numerical methods or software for more complex integrals involving specific functions.

In summary, analyzing the region D and choosing an appropriate coordinate transformation are crucial steps in evaluating double integrals over unbounded regions. The transformation $u = X / Y$, $v = Y$ offers a clear pathway to simplify the region and facilitate integration, especially when

Frequently Asked Questions

What is the region Uxy Da in the first quadrant bounded by the parabolas x = y and x = 8y?

The region Uxy Da is the area in the first quadrant between the two parabolas $x = y$ and $x = 8y$, where both x and y are positive, bounded by their intersection points.

How do you find the points of intersection between the parabolas x = y and x = 8y?

Set $y = 8y$, which simplifies to $y = 8y$, leading to $y = 0$ or y undefined; more accurately, equate $x = y$ and $x = 8y$, so $y = 8y$, which implies $y = 0$ (since $y \neq \text{infinity}$), and at $y=0$, $x=0$. Thus, both parabolas intersect at the origin (0,0).

What are the limits of integration when describing the region Uxy Da?

The limits of y are from 0 up to the intersection point where $x = y$ and $x = 8y$ meet; since they intersect at (0,0), and for $y > 0$, the region is between $y = 0$ and the line $x = 8y$, with x varying from y to 8y.

How can the region Uxy Da be described in terms of inequalities?

The region Uxy Da can be described by the inequalities: $0 \leq y \leq y_{max}$, and $y \leq x \leq 8y$, where y_{max} is determined by the intersection point of the parabolas in the first quadrant.

How do you compute the area of the region Uxy Da?

The area can be found by integrating with respect to x or y. For example, integrating y from 0 to y_{max} and x from y to 8y: Area = $\int_0^{y_{max}} \int_{x=y}^{8y} dx\, dy$, where y_{max} is the y-coordinate of the intersection point, which can be found by solving the equations simultaneously.

What is the significance of the region Uxy Da in calculus problems?

This region is often used as a domain for double integrals, helping to evaluate areas, volumes, or other quantities over regions bounded by curves, and serves as an example of setting up integrals with specific boundaries in the first quadrant.

Additional Resources

Uxy Da, Where D Is The Region In The First Quadrant Bounded By The Parabolas X = Y And X = 8 Y is a classic problem in multivariable calculus that illuminates the process of setting up and evaluating double integrals over complex regions. This region, defined by two intersecting parabolas in the first quadrant, offers an excellent example of how to analyze and compute integrals over bounded areas, especially when the boundaries are curves rather than simple rectangles. In this guide, we will walk through the detailed steps involved in understanding, visualizing, and evaluating the double integral over this region, providing a comprehensive resource for students and professionals alike.

Understanding the Region D in the First Quadrant

Before diving into the calculations, it’s essential to understand the region D itself. The problem specifies that D is the region in the first quadrant bounded by the parabolas:

- $X = Y$
- $X = 8Y$

Visualizing the Region

In the xy-plane:

- The line $X = Y$ is a straight line passing through the origin with a 45-degree slope.
- The line $X = 8Y$ is also a straight line passing through the origin but much steeper.

Since both are lines passing through the origin and we are restricted to the first quadrant (where $X \geq 0, Y \geq 0$), the region D is the sector bounded between these two lines for positive Y and X.

Finding the Intersection Points

The boundary lines intersect at the origin $(0,0)$. To identify the limits of the region, note:

- Along $Y = t$, the corresponding X values on the boundaries are:

- On $X = Y$, $X = t$.
- On $X = 8Y$, $X = 8t$.

- The two lines intersect only at the origin, so the region D extends from the origin outward between these lines.

Setting Up the Double Integral

When integrating over D, choosing the right order of integration (either integrating with respect to X first or Y first) simplifies the process.

Option 1: Integrate with respect to X first

- For a fixed Y, X varies between the lower boundary $X = Y$ and the upper boundary $X = 8Y$.

- The limits for Y need to be determined: how far does the region extend in the Y-direction?

Since both lines pass through the origin and extend outward, and the region is bounded between the lines in the first quadrant, the upper limit in Y can be derived from where the two lines intersect or from any other boundary.

But in this case, because both lines pass through the origin and extend infinitely, the region is unbounded unless specified with a finite limit or other boundary. However, in typical problems, we consider a bounded portion, or assume the integration is over the region between these lines from $Y=0$ up to a certain Y, or for the entire sector.

For now, assume the region extends from $Y=0$ to some $Y = Y_{max}$. If the problem specifies a finite region, it would be given; otherwise, the integral may be improper or considered over a bounded segment.

Final Setup:

$$
\iint_{D} f(x,y) \, dx\,dy = \int_{Y=0}^{Y_{max}} \int_{X=Y}^{8Y} f(x,y) \, dx\, dy
$$

Changing the Order of Integration: Variable Substitution

Sometimes, transforming the region into a more manageable shape simplifies the integration process.

Using the Lines as Limits

- $X = Y$ and $X = 8Y$ suggest that for a fixed X, the corresponding Y values satisfy:

$$
Y = X \quad \text{and} \quad Y = \frac{X}{8}
$$

- For X in the range from 0 to some maximum, Y varies between:

$$
Y = \frac{X}{8} \quad \text{and} \quad Y= X
$$

- The maximum X-value depends on the region considered.

Expressing the Region in Terms of X

Suppose we want to express the region with:

- X from 0 to a maximum X_{max},
- Y from $\frac{X}{8}$ to X.

The limits become:

$$
\iint_D f(x,y) \, dx\,dy = \int_{X=0}^{X_{max}} \int_{Y=\frac{X}{8}}^{Y=X} f(x,y) \, dy \, dx
$$

Transformation of Coordinates: A Strategic Approach

In some cases, especially with curves like parabolas, it can be advantageous to perform coordinate transformations to simplify the boundaries.

Parametric or Change of Variables

Given the boundaries:

- $X = Y$,
- $X = 8Y$,

we can consider a substitution:

$$
u = X, \quad v = \frac{X}{Y}

$$

which simplifies to:

$$
v = \frac{X}{Y}
$$

In the original boundaries:

- $v = 1$ corresponds to $X=Y$,
- $v=8$ corresponds to $X=8Y$.

This change maps the region D to the rectangle:

$$
u \in [0, \text{some } U], \quad v \in [1, 8]
$$

which can make integration more straightforward, especially if $f(x,y)$ becomes simpler in u,v variables.

Computing the Double Integral: Step-by-Step Guide

Let’s assume we want to compute a specific integral, say:

$$
I = \iint_{D} xy \, dx\, dy
$$

Step 1: Define the Region

- As established, the region D in the xy-plane is bounded between:

$$
Y = \frac{X}{8} \quad \text{and} \quad Y = X
$$

for X in some interval from 0 to X_{max}, which could be infinity or a finite boundary.

Step 2: Choose the Order of Integration

Let's choose to integrate with respect to Y first:

$$
I = \int_{X=0}^{X_{max}} \int_{Y=\frac{X}{8}}^{Y=X} xy \, dy \, dx
$$

Step 3: Integrate with respect to Y

- For fixed X:

$$
\int_{Y=\frac{X}{8}}^{Y=X} x y \, dy = x \int_{Y=\frac{X}{8}}^{Y=X} y \, dy = x \left[\frac{y^2}{2} \right]_{Y=\frac{X}{8}}^{Y=X}
$$

- Compute the inner integral:

$$
x \left(\frac{X^2}{2} - \frac{(X/8)^2}{2} \right) = x \left(\frac{X^2}{2} - \frac{X^2}{2 \times 64} \right) = x \left(\frac{X^2}{2} - \frac{X^2}{128} \right)
$$

$$
= x \times X^2 \left(\frac{1}{2} - \frac{1}{128} \right) = x X^2 \times \left(\frac{64}{128} - \frac{1}{128} \right) = x X^2 \times \frac{63}{128}
$$

Step 4: Integrate with respect to X

- The integral becomes:

$$
I = \frac{63}{128} \int_{0}^{X_{max}} x X^2 \, dx
$$

- Since x and X are the same variable, the integral simplifies to:

$$
I = \frac{63}{128} \int_{0}^{X_{max}} X^3 \, dX = \frac{63}{128} \left[\frac{X^4}{4} \right]_0^{X_{max}} = \frac{63}{128} \times \frac{X_{max}^4}{4}
$$

Step 5: Final Result

- The value of the integral depends on the upper limit X_{max}. If the region is unbounded, the integral diverges; otherwise, for a finite boundary, substitute the specific X_{max} to get a numerical result.

Special Considerations and Variations

Handling Infinite Boundaries

If the region D extends infinitely, the integral may be improper and require convergence tests. For instance, integrating functions like \(xy \) over an infinite region bounded by straight lines may or may not converge depending on the integrand.

Alternative

2 Uxy Da Where D Is The Region In The First Quadrant Bounded By The Parabolas X Y And X 8 Y

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