(20 Pts) Using The Definition Of The Asymptotic Notations, Show That A) 6n 2+n=(n 2) B) 6n 2=O(2n)

(20 Pts) Using The Definition Of The Asymptotic Notations, Show That A) 6n^2 + n = (n^2) B) 6n^2 = O(2^n)

Introduction

Asymptotic analysis is a crucial aspect of computer science, especially when evaluating the efficiency of algorithms. By understanding how an algorithm's running time or space requirement grows concerning the input size, developers and researchers can make informed decisions about which algorithms to implement in different scenarios. At the core of asymptotic analysis are asymptotic notations, which provide formal ways to describe the limiting behavior of functions as their input approaches infinity.

This article aims to thoroughly explore the concept of asymptotic notations—particularly Big O, Big Theta, and Big Omega—and demonstrate, through rigorous definitions, how to establish that:

- A) $6n^2 + n = \Theta(n^2)$
- B) $6n^2 = O(2^n)$

We will systematically approach each proof, emphasizing the importance of the formal definitions and illustrating the steps involved in these derivations.

Understanding Asymptotic Notations

What Are Asymptotic Notations?

Asymptotic notations are mathematical tools used to describe the behavior of functions as their input size, typically denoted by n, approaches infinity. They are essential for analyzing the theoretical performance of algorithms.

Types of Asymptotic Notations

1. Big O Notation (O): Describes an upper bound on the growth rate of a function.
- Interpretation: The function does not grow faster than a certain rate beyond some point.

2. Big Theta Notation (Θ): Provides a tight bound, indicating that a function grows at the same rate as another function, up to constant factors.

- Interpretation: The function is asymptotically bounded both above and below by the same function.

3. Big Omega Notation (Ω): Describes a lower bound on the growth rate of a function.
- Interpretation: The function grows at least as fast as a given rate beyond some point.

Formal Definitions

Let f(n) and g(n) be functions from natural numbers to positive real numbers.

- Big O (O):
$f(n) = O(g(n))$ if there exist positive constants c and n_0 such that for all $n \geq n_0$:
$$ f(n) \leq c \times g(n) $$

- Big Theta (Θ):
$f(n) = \Theta(g(n))$ if there exist positive constants c_1, c_2, and n_0 such that for all $n \geq n_0$:
$$ c_1 \times g(n) \leq f(n) \leq c_2 \times g(n) $$

- Big Omega (Ω):
$f(n) = \Omega(g(n))$ if there exist positive constants c and n_0 such that for all $n \geq n_0$:
$$ f(n) \geq c \times g(n) $$

Part A: Showing That $6n^2 + n = \Theta(n^2)$

Step 1: Recognizing the Function's Behavior

The function in question is:

$$ f(n) = 6n^2 + n $$

As n grows large, the term $6n^2$ dominates n. Intuitively, $f(n)$ behaves like n^2, suggesting that:

$$ 6n^2 + n = \Theta(n^2) $$

But to formally prove this, we need to apply the definition of Big Theta.

Step 2: Applying the Definition of Θ-notation

To show that:

$$ 6n^2 + n = \Theta(n^2) $$

we must find positive constants c_1, c_2, and n_0 such that:

$$ c_1 \times n^2 \leq 6n^2 + n \leq c_2 \times n^2 \quad \text{for all } n \geq n_0 $$

Step 3: Establishing the Upper Bound

Find c_2 such that:

$$ 6n^2 + n \leq c_2 \times n^2 $$

Dividing both sides by n^2 (for $n \geq 1$):

$$ 6 + \frac{1}{n} \leq c_2 $$

Since $\frac{1}{n}$ decreases as n increases, for all $n \geq 1$:

$$ 6 + \frac{1}{n} \leq 6 + 1 = 7 $$

Thus, choosing:

$$ c_2 = 7 $$

and

$$ n_0 = 1 $$

we have, for all $n \geq 1$:

$$ 6n^2 + n \leq 7 n^2 $$

Step 4: Establishing the Lower Bound

Find c_1 such that:

$$ c_1 \times n^2 \leq 6n^2 + n $$

Dividing both sides by n^2:

$$ c_1 \leq 6 + \frac{1}{n} $$

As n approaches infinity, $\frac{1}{n} \to 0$. For all sufficiently large n, say $n \geq n_0$, the term:

$$ 6 + \frac{1}{n} \geq 6 $$

Thus, choosing:

$$ c_1 = 6 $$

and

$$ n_0 = 1 $$

for all $n \geq 1$:

$$ 6n^2 + n \geq 6 n^2 $$

Step 5: Finalizing the Proof

Having established:

- For all $n \geq 1$:

$$ 6 n^2 \leq 6n^2 + n \leq 7 n^2 $$

we conclude:

$$ c_1 = 6, \quad c_2 = 7, \quad n_0 = 1 $$

Therefore,

$$ 6n^2 + n = \Theta(n^2) $$

This demonstrates that the function's growth rate is tightly bound by constant multiples of n^2, confirming the asymptotic equivalence.

Part B: Showing That $6n^2 = O(2^n)$

Step 1: Recognizing the Growth Rates

The function:

$$ f(n) = 6n^2 $$

is polynomial, while:

$$ g(n) = 2^n $$

is exponential. Intuitively, exponential functions grow faster than polynomial functions as n approaches infinity. To formally prove:

$$ 6n^2 = O(2^n) $$

we need to find constants c and n_0 such that:

$$ 6n^2 \leq c \times 2^n \quad \text{for all } n \geq n_0 $$

Step 2: Applying the Big O Definition

Let’s choose specific values to find suitable c and n_0.

Step 3: Finding Suitable Constants

Observe that as n increases:

- $6n^2$ grows quadratically.
- 2^n grows exponentially.

To find c, consider the ratio:

$$\frac{6n^2}{2^n}$$

As $n \to \infty$, this ratio approaches zero, implying that $6n^2$ is asymptotically dominated by 2^n.

Let's pick some values of n:

n	$6n^2$	2^n	$\frac{6n^2}{2^n}$
10	6100=600	1024	~0.585
15	6225=1350	32768	~0.041
20	6400=2400	1,048,576	~0.0023
25	6625=3750	33,554,432	~0.000111

From the table, we see that for $n \geq 10$:

$$6n^2 \leq c \times 2^n$$

Choosing $c = 1$, for $n \geq 10$:

$$6n^2 \leq 2^n$$

since the ratio is less than 1 and decreasing.

Step 4: Formal Selection of Constants

- Set:

$$c = 1$$
- Choose:

$$n_0 = 10$$

then, for all $n \geq 10$:

$$6n^2 \leq 2^n$$

which confirms:

$$6n^2 = O(2^n)$$

Step

Frequently Asked Questions

How do we formally define the asymptotic notation $O(\cdot)$?

The Big O notation $O(g(n))$ describes an upper bound on a function $f(n)$, meaning there exist positive constants c and n_0 such that for all $n \geq n_0$, $|f(n)| \leq c \cdot |g(n)|$. It formalizes the idea of the growth rate of functions.

How can we prove that $6n^2 + n = O(n^2)$ using the definition of Big O notation?

To prove $6n^2 + n = O(n^2)$, choose $c = 7$ and $n_0 = 1$. For all $n \geq n_0$, $|6n^2 + n| \leq 6n^2 + n \leq 6n^2 + n^2 = 7n^2$. Since $c = 7$ satisfies the inequality, $6n^2 + n = O(n^2)$.

What is the reasoning behind showing that $6n^2 + n = \Theta(n^2)$?

Because $6n^2 + n$ is both $O(n^2)$ and $\Omega(n^2)$, with constants $c_1 = 6$ and $c_2 = 7$, and for sufficiently large n, the function behaves proportionally to n^2, confirming that $6n^2 + n = \Theta(n^2)$.

How do we demonstrate that $6n^2 = O(2^n)$ using the definition of asymptotic notation?

Since polynomial functions grow slower than exponential functions, for all $n \geq 1$, $6n^2 \leq 6 \cdot 2^n$. Choosing $c = 6$ and $n_0 = 1$, we get $|6n^2| \leq c \cdot |2^n|$ for all $n \geq n_0$, proving $6n^2 = O(2^n)$.

Why is it correct to say that $6n^2 = O(2^n)$, but not vice versa?

Because polynomial functions like $6n^2$ grow asymptotically slower than exponential functions like 2^n, making $6n^2 = O(2^n)$. Conversely, 2^n is not $O(n^2)$ because exponential growth outpaces polynomial growth, so the relation does not hold in reverse.

Additional Resources

(20 Pts) Using The Definition Of The Asymptotic Notations, Show That A) $6n^2 + n = (n^2)$ B) $6n^2 = O(2^n)$

Introduction

Understanding the efficiency of algorithms is a cornerstone of computer science, particularly when it comes to analyzing how well an algorithm performs as the input size grows large. To facilitate this analysis, computer scientists employ asymptotic notations, such as Big O, Big Theta, and Big Omega, which provide a high-level understanding of an algorithm's growth rate. These notations help in comparing algorithms and predicting their performance without getting bogged down in the specifics of hardware or implementation details.

In this article, we will delve into the fundamental definitions of asymptotic notations and use these definitions to demonstrate two specific mathematical statements:

A) $6n^2 + n = \Theta(n^2)$

B) $6n^2 = O(2^n)$

Through a detailed, step-by-step approach, we aim to clarify how these relationships are established and what they imply about the growth behaviors of the functions involved.

Foundations: What Are Asymptotic Notations?

Before diving into proofs, it’s essential to understand the core definitions.

Big O Notation (O)

Big O notation provides an upper bound on the growth rate of a function. Formally:

Definition:
A function $f(n)$ is said to be $O(g(n))$ if there exist positive constants c and n_0 such that for all $n \geq n_0$:

$$
f(n) \leq c \cdot g(n)
$$

This means that beyond some threshold n_0, $f(n)$ does not grow faster than a constant multiple of $g(n)$.

Big Theta Notation (Θ)

Big Theta provides a tight bound:

Definition:
A function $f(n)$ is $\Theta(g(n))$ if there are positive constants

c_1, c_2, and n_0 such that for all $n \geq n_0$:

$$
c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)
$$

This indicates that $f(n)$ grows asymptotically at the same rate as $g(n)$.

Part A: Showing $6n^2 + n = \Theta(n^2)$

Let’s analyze the statement:

$$
6n^2 + n = \Theta(n^2)
$$

This is a claim about the equivalence in growth rate between the polynomial $6n^2 + n$ and n^2.

Step 1: Understanding the Intuition

As n becomes very large, the dominant term in $6n^2 + n$ is $6n^2$. The linear term n becomes insignificant compared to the quadratic term because:

$$
\lim_{n \to \infty} \frac{n}{n^2} = 0
$$

Hence, the entire function behaves like a constant multiple of n^2 for large n.

Step 2: Formal Proof Using Definitions

To prove $6n^2 + n = \Theta(n^2)$, we need to find positive constants c_1, c_2, n_0 such that for all $n \geq n_0$:

$$
c_1 n^2 \leq 6n^2 + n \leq c_2 n^2
$$

Upper Bound:

Find c_2 and n_0 such that:

$$
6n^2 + n \leq c_2 n^2
$$

Dividing through by n^2:

$$
6 + \frac{1}{n} \leq c_2
$$

As n grows large, $\frac{1}{n}$ approaches 0. To ensure the inequality holds for all $n \geq n_0$, choose n_0 such that:

$$
\frac{1}{n_0} \leq 1
$$

which implies $n_0 \geq 1$. Let's pick $n_0 = 1$. Then, for all $n \geq 1$:

$$
6 + \frac{1}{n} \leq 6 + 1 = 7
$$

So, for all $n \geq 1$, we have:

$$
6n^2 + n \leq 7 n^2
$$

Thus, $c_2 = 7$.

Lower Bound:

Find c_1 and n_0 such that:

$$
c_1 n^2 \leq 6n^2 + n
$$

Dividing through by n^2:

$$
c_1 \leq 6 + \frac{1}{n}
$$

As $n \to \infty$, $\frac{1}{n} \to 0$. To be safe, pick $n_0 = 1$:

$$
c_1 \leq 6 + 1 = 7
$$

To get a strict lower bound, choose $c_1 = 6$. Then, for all $n \geq 1$:

\[

$$
6 n^2 \leq 6 n^2 + n
$$

since $n \geq 1$, $n \leq n^2$, so:

$$
6 n^2 \leq 6 n^2 + n
$$

Therefore, the inequalities are satisfied with:

- $c_1 = 6$,
- $c_2 = 7$,
- $n_0 = 1$.

Conclusion:

$$
6n^2 + n = \Theta(n^2)
$$

This formal proof demonstrates that for sufficiently large n, the function $6n^2 + n$ is tightly bound by constants times n^2, confirming the asymptotic equivalence.

Part B: Showing $6n^2 = O(2^n)$

Now, let's analyze the second statement:

$$
6n^2 = O(2^n)
$$

This indicates that the quadratic function $6n^2$ grows at most as fast as an exponential function 2^n.

Step 1: Understanding the Intuition

Exponential functions grow much faster than polynomial functions. As n increases, 2^n outpaces any polynomial, including $6n^2$. Therefore, it should be straightforward to establish an inequality that bounds $6n^2$ above by some constant multiple of 2^n.

Step 2: Formal Proof Using Definition

To demonstrate $6n^2 = O(2^n)$, we need to find positive constants c and n_0 such that:

\[

6n^2 \leq c \cdot 2^n, \quad \forall n \geq n_0
\]

Choosing constants:

Since \(2^n\) grows rapidly, for small \(n\), the inequality may not hold. But for sufficiently large \(n\), it will.

Let's analyze the ratio:

\[
\frac{6 n^2}{2^n}
\]

As \(n \to \infty\), this ratio approaches 0 because exponential growth dominates polynomial growth. To find a specific \(n_0\), we can test some values:

- For \(n = 10\):

\[
6 \times 10^2 = 600
\]
\[
2^{10} = 1024
\]
\[
\frac{600}{1024} \approx 0.5859
\]

So, \(6 n^2 \leq 2^{n}\) for \(n \geq 10\).

- For \(n=5\):

\[
6 \times 25 = 150
\]
\[
2^5 = 32
\]
\[
\frac{150}{32} \approx 4.6875
\]

Here, \(6n^2\) exceeds \(2^n\), so the inequality doesn't hold at \(n=5\).

- For \(n=8\):

\[
6 \times 64 = 384
\]

$$
2^8 = 256
$$

$$
\frac{384}{256} = 1.5
$$

Still not satisfying $6 n^2 \leq c \cdot 2^n$ for a fixed constant c when $n=8$, unless $c \geq 1.5$.

At $n=10$, the ratio is less than 1, so the inequality:

$$
6 n^2 \leq 2^n
$$

holds with $c=1$ for $n \geq 10$.

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