42. Find The Equation Of The Sphere With Center C(2,3,7) That Is Tangent To The Plane 2x+3y6z=5.

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Understanding how to determine the equation of a sphere given specific conditions is a fundamental concept in three-dimensional geometry. In this article, we'll explore the process of finding the equation of a sphere with a known center, C(2, 3, 7), that is tangent to a given plane, specifically 2x + 3y + 6z = 5. This problem combines concepts of the distance from a point to a plane, the properties of tangent spheres, and the algebraic formulation of a sphere's equation.

Introduction to Sphere Equations and Tangency Concepts

A sphere in three-dimensional space is defined by its center and radius. The standard form of a sphere's equation with center C(h, k, l) and radius r is:

$$ (x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 $$

The problem here involves finding the specific sphere that:

- Has a center at C(2, 3, 7), and
- Is tangent to the plane 2x + 3y + 6z = 5.

Tangency means the sphere touches the plane at exactly one point, implying that the shortest distance from the center of the sphere to the plane equals the radius of the sphere.

Step-by-Step Approach to Find the Equation of the Sphere

To determine the equation of the sphere, we'll follow these main steps:

1. Identify the sphere's center: Given as C(2, 3, 7).

2. Calculate the distance from the center to the plane: This distance will be equal to the radius since the sphere is tangent to the plane.

3. Determine the radius: Using the distance calculated in step 2.

4. Write the sphere's equation: Using the center coordinates and radius.

Let's examine each step meticulously.

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1. Understanding the Given Data

The center C(2, 3, 7) is straightforward, and the plane equation is:

$$ 2x + 3y + 6z = 5 $$

This plane can be rewritten in standard form as:

$$ 2x + 3y + 6z - 5 = 0 $$

The coefficients in the plane equation are:

- $A = 2$,
- $B = 3$,
- $C = 6$,
- $D = -5$.

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2. Calculating the Distance from the Center to the Plane

The shortest distance d from a point $P(x_0, y_0, z_0)$ to a plane $Ax + By + Cz + D = 0$ is given by:

$$
d = \frac{|A x_0 + B y_0 + C z_0 + D|}{\sqrt{A^2 + B^2 + C^2}}
$$

Substituting $P(2, 3, 7)$ and the plane coefficients:

$$
d = \frac{|2 \times 2 + 3 \times 3 + 6 \times 7 - 5|}{\sqrt{2^2 + 3^2 + 6^2}}
$$

Compute numerator:

$$
2 \times 2 = 4
$$

$$
3 \times 3 = 9
$$
$$
6 \times 7 = 42
$$

Sum:

$$
4 + 9 + 42 = 55
$$

Subtract 5:

$$
55 - 5 = 50
$$

Absolute value:

$$
|50| = 50
$$

Compute denominator:

$$
\sqrt{4 + 9 + 36} = \sqrt{49} = 7
$$

Therefore,

$$
d = \frac{50}{7}
$$

This distance is the radius r of the sphere since the sphere is tangent to the plane:

$$
r = \frac{50}{7}
$$

---

3. Formulating the Equation of the Sphere

Now that the radius $r = \frac{50}{7}$ and the center $C(2, 3, 7)$ are known, the equation of the sphere is:

$$
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = r^2
$$

Calculate r^2:

$$
r^2 = \left(\frac{50}{7}\right)^2 = \frac{2500}{49}
$$

Thus, the equation becomes:

$$
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = \frac{2500}{49}
$$

Final Equation of the Sphere

Answer:

$$
\boxed{
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = \frac{2500}{49}
}
$$

This is the equation of the sphere with center at (2, 3, 7) that is tangent to the plane 2x + 3y + 6z = 5.

Additional Insights and Applications

Understanding the derivation and formulation of the sphere's equation has numerous applications in geometry, physics, and engineering. Some key insights include:

- Tangency condition: The key property used here is that the shortest distance from the sphere's center to the tangent plane equals the radius.

- Geometric interpretation: The sphere "touches" the plane at exactly one point, which is the point of tangency.

- Real-world applications:
- Designing spherical objects that fit snugly against flat surfaces.

- Calculating the minimum distance between a sphere and a plane in collision detection.
- Optimizing spatial arrangements in 3D modeling.

Related Concepts and Extensions

For a more comprehensive understanding, consider exploring the following topics:

- Sphere equations with different centers and radii.
- Distance from a point to a plane, including cases when the sphere is not tangent but intersects or is disjoint.
- Tangent spheres to multiple planes or surfaces.
- Sphere and plane intersections, leading to circle equations.
- Sphere with a given radius passing through a specific point.

Summary

To summarize, the process of finding the equation of a sphere with a specified center and tangent to a given plane involves:

- Calculating the distance from the center to the plane.
- Recognizing that this distance equals the radius for tangency.
- Substituting the center and radius into the standard sphere equation.

In this specific case, the sphere with center $C(2, 3, 7)$ tangent to $2x + 3y + 6z = 5$ has the equation:

$$
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = \frac{2500}{49}
$$

This formula provides a precise mathematical description of the sphere satisfying the given conditions, serving as a foundational example in three-dimensional analytic geometry.

Note: Mastery of these concepts enhances spatial visualization skills and problem-solving capabilities in advanced geometry and related fields.

Frequently Asked Questions

How do you find the equation of a sphere with a given center that is tangent to a specific plane?

To find the sphere's equation, you identify the center coordinates and determine the radius as the perpendicular distance from the center to the plane, since the sphere is tangent to the plane. The equation is then $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

What is the center of the sphere in this problem?

The center of the sphere is C(2, 3, 7).

How do you compute the radius of the sphere given its tangency to the plane?

The radius is equal to the perpendicular distance from the sphere's center to the plane. This distance is calculated using the formula: $|A x_0 + B y_0 + C z_0 + D| / \sqrt{(A^2 + B^2 + C^2)}$.

What is the equation of the plane in standard form?

The plane's equation is $2x + 3y + 6z = 5$. It is already in standard form: A=2, B=3, C=6, D=-5.

How do you calculate the perpendicular distance from the center to the plane?

Use the formula: $|A x_0 + B y_0 + C z_0 + D| / \sqrt{(A^2 + B^2 + C^2)}$. Plugging in the values gives the distance, which is the sphere's radius.

Calculate the distance from the center to the plane for this problem.

Distance = $|22 + 33 + 67 - 5| / \sqrt{(2^2 + 3^2 + 6^2)} = |4 + 9 + 42 - 5| / \sqrt{(4 + 9 + 36)} = |50| / \sqrt{49} = 50 / 7$.

What is the radius of the sphere in this problem?

The radius r = 50/7.

What is the equation of the sphere with center C(2, 3, 7) and radius 50/7?

The equation is $(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = (50/7)^2$.

Why is the sphere's radius equal to the distance from the center to the plane?

Because the sphere is tangent to the plane, the point of tangency lies on both the sphere and the plane, making the radius equal to the perpendicular distance from the center to the plane.

Can you write the final equation of the sphere explicitly?

Yes, the equation is $(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = (50/7)^2$, which simplifies to $(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = 2500/49$.

Additional Resources

42. Find The Equation Of The Sphere With Center C(2,3,7) That Is Tangent To The Plane 2x+3y+6z=5

Introduction

In the realm of three-dimensional geometry, the problem of determining the equation of a sphere based on given parameters such as its center and tangency conditions to a plane stands as a fundamental exercise with broad applications—ranging from engineering design to computer graphics. The problem statement: "Find the equation of a sphere with center C(2,3,7) that is tangent to the plane 2x + 3y + 6z = 5" is a classic example illustrating the interplay between geometric constructs and algebraic formulations.

This article offers an in-depth exploration of the problem, emphasizing the mathematical principles, step-by-step problem-solving techniques, and potential extensions. It aims to serve as a comprehensive resource for students, educators, and professionals seeking to understand and apply these concepts.

Mathematical Background and Concepts

The Equation of a Sphere

A sphere in three-dimensional space, with center $C(h, k, l)$ and radius r, is described algebraically by the equation:

$$
(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2
$$

Given the center $C(2, 3, 7)$, the general form becomes:

$$
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = r^2
$$

The task reduces to determining the radius r such that the sphere is tangent to the given plane.

Tangency to a Plane

A sphere is tangent to a plane if and only if the shortest distance from the sphere's center to the plane equals the radius of the sphere. This is a crucial geometric principle that simplifies the problem:

rather than explicitly solving for the intersection points, we focus on the distance from the center to the plane.

Step-by-Step Solution

Step 1: Write the Equation of the Plane

Given the plane:

$$
2x + 3y + 6z = 5
$$

which can be rewritten in the standard form:

$$
2x + 3y + 6z - 5 = 0
$$

The coefficients of x, y, z give the normal vector to the plane:

$$
\mathbf{n} = \langle 2, 3, 6 \rangle
$$

Step 2: Calculate the Distance from the Center to the Plane

The distance d from a point $C(h, k, l)$ to the plane $Ax + By + Cz + D = 0$ is given by:

$$
d = \frac{|A h + B k + C l + D|}{\sqrt{A^2 + B^2 + C^2}}
$$

Substituting the known values:

- $A = 2$, $B = 3$, $C = 6$, $D = -5$
- $h = 2$, $k = 3$, $l = 7$

Calculate numerator:

$$
|2 \times 2 + 3 \times 3 + 6 \times 7 - 5| = |4 + 9 + 42 - 5| = |50| = 50
$$

Calculate denominator:

$$
\sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7
$$

Thus,

$$
d = \frac{50}{7}
$$

Step 3: Determine the Radius

Since the sphere is tangent to the plane, the radius r equals the distance from the center to the plane:

$$
r = d = \frac{50}{7}
$$

Final Equation of the Sphere

Replacing the radius into the sphere's equation:

$$
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = \left(\frac{50}{7}\right)^2
$$

Expressed explicitly:

$$
(x - 2)^2 + (y - 3)^2 + (z - 7)^2 = \frac{2500}{49}
$$

This is the equation of the sphere with center $C(2, 3, 7)$ tangent to the plane $2x + 3y + 6z = 5$.

Geometric Insights and Significance

The Role of the Normal Vector

The normal vector $\mathbf{n} = \langle 2, 3, 6 \rangle$ is perpendicular to the plane and points in the direction of the shortest distance from the center to the plane. The magnitude of this vector influences the calculation of the distance and, consequently, the radius.

Tangency and Distance

The core geometric principle employed here is that tangency occurs when the distance from the sphere's center to the tangent plane equals the sphere's radius. This provides a straightforward method to determine the radius without solving for intersection points explicitly.

Extensions and Related Problems

1. Sphere Tangent to Multiple Planes

Given multiple planes, finding a sphere tangent to all involves solving systems where the distances from the center to each plane equal respective radii, potentially leading to more complex systems.

2. Sphere Passing Through a Point and Tangent to a Plane

Suppose the sphere passes through a specific point $P(x_p, y_p, z_p)$ and is tangent to a plane. The problem involves setting up equations based on the distance and point conditions.

3. Sphere with Given Radius and Center

Conversely, if the radius is given, one can verify tangency conditions or determine whether the sphere touches certain geometric objects.

Practical Applications

- Engineering and Design: Ensuring components fit precisely, with spheres tangent to surfaces.
- Computer Graphics: Rendering spheres that interact with planes for visual realism.
- Robotics and Navigation: Calculating safe zones or sensor ranges modeled as spheres tangent to surfaces.

Concluding Remarks

The systematic approach to this problem demonstrates the power of geometric principles integrated with algebraic techniques. By understanding the relationship between a sphere's center, radius, and tangent planes, we unlock a versatile toolkit applicable across various scientific and engineering disciplines.

The explicit derivation of the sphere's equation offers clarity and precision, serving as a foundational example for students and practitioners alike. In more advanced contexts, these principles extend naturally to more complex surfaces and higher-dimensional analogs, underscoring their fundamental importance.

References

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This investigation showcases the elegance of geometric reasoning and algebraic computation in deriving precise equations for three-dimensional shapes, reinforcing the interconnected nature of mathematical concepts.

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