An Implicit Equation For The Plane Passing Through The Points (4,1,3), (6,2,8), And (1,1,2) Is

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Understanding the geometric properties of planes in three-dimensional space is fundamental in fields such as mathematics, engineering, computer graphics, and physics. One of the core tasks is to determine the explicit or implicit equations of planes passing through given points. This guide provides a comprehensive explanation of how to derive the implicit equation of a plane passing through three specific points: (4, 1, 3), (6, 2, 8), and (1, 1, 2). By the end of this article, you'll understand the step-by-step process to find such an equation, the underlying mathematical principles, and practical applications.

Understanding Planes in 3D Space

The Basics of Plane Equations

In three-dimensional space, a plane can be represented in various forms:

- **Explicit form:** y = mx + bz + c (less common for planes)
- **Point-normal form:** $(X - X_0)\cdot n = 0$, where n is a normal vector to the plane, and $(X - X_0)$ is the vector from a point on the plane to any arbitrary point X.
- **Implicit form:** Ax + By + Cz + D = 0, where A, B, C, D are constants, and (x, y, z) is any point on the plane.

The implicit form is particularly useful because it directly encodes the plane's orientation and position in space.

Why Use the Implicit Equation?

The implicit equation is beneficial because:

- It provides a straightforward way to test if a point lies on the plane.
- It facilitates intersection calculations with other geometric objects.
- It simplifies the process of deriving the plane equation from three points.

Step-by-Step Derivation of the Plane Equation

To find the implicit equation of the plane passing through three non-collinear points, follow these steps:

1. Verify the Points Are Non-Collinear

Ensure that the points (4, 1, 3), (6, 2, 8), and (1, 1, 2) are not collinear. If they are collinear, they won't define a unique plane.

Calculate vectors:

- Vector AB: from point A (4, 1, 3) to point B (6, 2, 8):

$$
\vec{AB} = (6 - 4, 2 - 1, 8 - 3) = (2, 1, 5)
$$

- Vector AC: from point A (4, 1, 3) to point C (1, 1, 2):

$$
\vec{AC} = (1 - 4, 1 - 1, 2 - 3) = (-3, 0, -1)
$$

Calculate the cross product $\vec{AB} \times \vec{AC}$, which should be non-zero if the points are not collinear.

2. Find the Normal Vector to the Plane

The normal vector $\vec{n}$ of the plane is perpendicular to both $\vec{AB}$ and $\vec{AC}$:

$$
\vec{n} = \vec{AB} \times \vec{AC}
$$

Calculate the cross product:

$$
\vec{n} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 1 & 5 \\
-3 & 0 & -1 \\
\end{vmatrix}
$$

Compute the determinant:

$$
\vec{n} = \mathbf{i}(1 \times -1 - 5 \times 0) - \mathbf{j}(2 \times -1 - 5 \times -3) + \mathbf{k}(2 \times 0 - 1 \times -3)
$$

Simplify each component:

- For $\mathbf{i}$:

$$
1 \times -1 - 5 \times 0 = -1 - 0 = -1
$$

- For $\mathbf{j}$:

$$
2 \times -1 - 5 \times -3 = -2 - (-15) = -2 + 15 = 13
$$

- For $\mathbf{k}$:

$$
2 \times 0 - 1 \times -3 = 0 + 3 = 3
$$

Therefore:

$$
\boxed{
\vec{n} = (-1, -13, 3)
}
$$

This vector is normal to the plane passing through the points.

3. Write the Equation of the Plane

Using the point-normal form:

$$
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0
$$

Where:

- (x_0, y_0, z_0) is any point on the plane, for example, (4, 1, 3).
- $\vec{n} = (A, B, C) = (-1, -13, 3)$.

Substitute the values:

$$
-1(x - 4) - 13(y - 1) + 3(z - 3) = 0
$$

Expand:

$$
-1x + 4 - 13y + 13 + 3z - 9 = 0
$$

Combine constants:

$$
-1x - 13y + 3z + (4 + 13 - 9) = 0
$$

$$
-1x - 13y + 3z + 8 = 0
$$

Multiplying through by -1 to make the leading coefficient positive:

$$
x + 13y - 3z - 8 = 0
$$

This is the implicit equation of the plane passing through the points (4, 1, 3), (6, 2, 8), and (1, 1, 2).

Interpreting the Equation

The implicit equation:

$$
x + 13y - 3z - 8 = 0
$$

encodes all points (x, y, z) lying on the plane. The coefficients directly relate to the plane's orientation, with the normal vector being $\mathbf{n} = (1, 13, -3)$.

Geometric Significance of the Equation

- The coefficients $A=1$, $B=13$, and $C=-3$ form the normal vector to the plane.
- The constant term -8 shifts the plane relative to the origin.
- The plane's position can be visualized by plugging in specific values.

Practical Applications of the Derived Plane Equation

Understanding the implicit equation of a plane has multiple applications:

- **Computer Graphics:** To render objects and perform clipping operations.
- **Engineering Design:** For modeling surfaces and analyzing structural components.
- **Robotics:** To navigate and avoid obstacles in 3D space.
- **Geographical Information Systems (GIS):** To model terrain and surface features.
- **Physics:** For calculating intersections of planes with other objects or fields.

Summary of Key Steps

To recap, here are the essential steps to find the implicit equation of a plane through three points:

1. Verify the points are not collinear by checking the cross product of vectors between them.
2. Calculate vectors between the points to determine the direction vectors.
3. Compute the cross product to find the normal vector of the plane.
4. Use the point-normal form to derive the implicit equation.
5. Simplify the equation to standard form: Ax + By + Cz + D = 0.

Additional Tips and Considerations

- Collinearity Check: Always verify the points are non-collinear; otherwise, the plane is undefined or degenerate.
- Normalization: Sometimes, normal vectors are normalized to unit length for consistency.
- Multiple Points: The method extends to more points by ensuring they satisfy the derived plane equation.
- Numerical Precision: Be cautious of floating-point precision errors, especially with very close points.

Conclusion

Deriving the implicit equation of a plane passing through three points involves a clear understanding of vector operations, particularly the cross product, and the point-normal form of a plane's equation. The process starts with verifying the points' non-collinearity, computing the normal vector, and then formulating the equation. The specific points (4, 1, 3), (6, 2, 8), and (1, 1,

Frequently Asked Questions

How do you find the implicit equation of a plane passing through three given points?

To find the implicit equation of a plane passing through three points, first determine two vectors lying on the plane by subtracting the coordinates of the points, then find their cross product to get the normal vector. Using the normal vector and one point, substitute into the plane equation form n·(r - r0) = 0 to obtain the implicit equation.

What is the step-by-step process to derive the plane equation passing through (4,1,3), (6,2,8), and (1,1,2)?

First, compute two vectors: v1 = (6-4, 2-1, 8-3) = (2, 1, 5) and v2 = (1-4, 1-1, 2-3) = (-3, 0, -1). Next, find their cross product to get the normal vector n. Then, write the plane equation using the point-normal form: n·(r - r0) = 0, substituting one of the points. Simplify to get the implicit equation.

What is the explicit implicit equation of the plane passing through the given points?

The implicit equation of the plane passing through points (4,1,3), (6,2,8), and (1,1,2) is 3x + 9y - 2z + 1 = 0.

Why is the cross product important in finding the implicit equation of a plane?

The cross product of two vectors lying on the plane gives a normal vector perpendicular to the plane. This normal vector is essential for forming the plane's equation in the implicit form, as it defines the orientation of the plane.

Can the points (4,1,3), (6,2,8), and (1,1,2) be collinear, and how does that affect the plane equation?

No, these points are not collinear; they do not lie on a single straight line. If they were collinear, they wouldn't define a unique plane, and the plane equation would be undefined or degenerate. Since they are not collinear, a unique plane passing through all three exists.

What are common mistakes to avoid when deriving the plane equation from three points?

Common mistakes include incorrectly computing the vectors, miscalculating the cross product, mixing coordinate components, or algebraic errors when substituting into the plane equation. Double-check calculations and ensure the vectors are correctly derived from the given points.

Additional Resources

An Implicit Equation For The Plane Passing Through The Points (4,1,3), (6,2,8), And (1,1,2)

Understanding how to derive the implicit equation of a plane passing through three distinct points is a fundamental skill in vector calculus and analytic geometry. The problem of determining an implicit equation for the plane passing through the points (4,1,3), (6,2,8), and (1,1,2) is a classic example that combines spatial reasoning with algebraic manipulation. This article aims to guide you through the process step-by-step, providing clear explanations and practical tips to master the concept.

Introduction to Plane Equations in 3D Space

In three-dimensional space, the equation of a plane can be expressed in various forms, with the most common being the implicit form:

$$ Ax + By + Cz + D = 0 $$

where A, B, and C are the components of the normal vector to the plane, and D is a scalar offset.

Given three non-collinear points, the goal is to find this equation—specifically, the coefficients A, B, C, and D—that satisfy all three points.

Step 1: Verifying the Points Are Non-Collinear

Before proceeding, ensure that the points are not collinear (i.e., lying on a single straight line). If they were collinear, no unique plane passes through all three.

Let's analyze the points:

- $P_1 = (4, 1, 3)$
- $P_2 = (6, 2, 8)$
- $P_3 = (1, 1, 2)$

Calculate vectors:

$$
\vec{P_1 P_2} = (6 - 4, 2 - 1, 8 - 3) = (2, 1, 5)
$$

\]
\[
\vec{P_1 P_3} = (1 - 4, 1 - 1, 2 - 3) = (-3, 0, -1)
\]

Check if these vectors are linearly independent:

\[
\text{Are they scalar multiples?}
\]

\[
\text{No, since } (2,1,5) \neq k \times (-3,0,-1) \text{ for any scalar }k.
\]

Therefore, the points are non-collinear, ensuring a unique plane exists.

Step 2: Find Two Direction Vectors

To determine the plane, find two vectors lying on the plane:

\[
\vec{v_1} = \vec{P_1 P_2} = (2, 1, 5)
\]
\[
\vec{v_2} = \vec{P_1 P_3} = (-3, 0, -1)
\]

These vectors will help us find the plane's normal vector.

Step 3: Computing the Normal Vector

The normal vector \(\vec{n}\) to the plane is perpendicular to both \(\vec{v_1}\) and \(\vec{v_2}\).
The cross product of these vectors gives \(\vec{n}\):

\[
\vec{n} = \vec{v_1} \times \vec{v_2}
\]

Calculating:

\[
\vec{n} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 1 & 5 \\
-3 & 0 & -1

\end{vmatrix}
\]

Breaking down the determinant:

\[
\vec{n} = \mathbf{i} (1 \times -1 - 5 \times 0) - \mathbf{j} (2 \times -1 - 5 \times -3) + \mathbf{k} (2 \times 0 - 1 \times -3)
\]

Simplify each component:

- For \(\mathbf{i}\):

\[
1 \times -1 - 5 \times 0 = -1 - 0 = -1
\]

- For \(\mathbf{j}\):

\[
2 \times -1 - 5 \times -3 = -2 - (-15) = -2 + 15 = 13
\]

- For \(\mathbf{k}\):

\[
2 \times 0 - 1 \times -3 = 0 + 3 = 3
\]

Therefore, the normal vector:

\[
\boxed{
\vec{n} = (-1, -13, 3)
}
\]

Step 4: Formulating the Plane Equation

The general implicit form:

\[
A x + B y + C z + D = 0
\]

with \(\vec{n} = (A, B, C) = (-1, -13, 3)\).

To find \(D\), plug in one of the points, say \(P_1 = (4, 1, 3)\):

$$
-1 \times 4 + (-13) \times 1 + 3 \times 3 + D = 0
$$

Calculate:

$$
-4 - 13 + 9 + D = 0
$$
$$
(-4 - 13 + 9) + D = 0
$$
$$
(-8) + D = 0
$$
$$
D = 8
$$

Thus, the implicit equation of the plane is:

$$
-1 x - 13 y + 3 z + 8 = 0
$$

or, multiplying through by -1 for simplicity:

$$
x + 13 y - 3 z - 8 = 0
$$

Final Equation of the Plane

$$
\boxed{
x + 13 y - 3 z - 8 = 0
}
$$

Additional Insights and Considerations

1. Verifying the Equation

Always check that the other points satisfy the derived equation:

- For $P_2 = (6, 2, 8)$:

$$
6 + 13 \times 2 - 3 \times 8 - 8 = 6 + 26 - 24 - 8 = 32 - 32 = 0
$$

- For $P_3 = (1, 1, 2)$:

$$
1 + 13 \times 1 - 3 \times 2 - 8 = 1 + 13 - 6 - 8 = 14 - 14 = 0
$$

Both points satisfy the equation, confirming its correctness.

2. Interpreting the Plane's Normal Vector

The normal vector $\vec{n} = (1, 13, -3)$ indicates the plane's orientation in space. Its components reflect the plane's tilt along the x, y, and z axes.

3. Applications of the Implicit Equation

This implicit representation is useful in:

- Calculating distances from points to the plane
- Determining intersections with other geometric objects
- Performing projections and transformations

Summary and Key Takeaways

- To find the implicit equation of a plane passing through three points:

1. Verify points are non-collinear
2. Find vectors between points
3. Calculate the normal vector via the cross product
4. Use a point to solve for D

- The process hinges on understanding vector operations and their geometric interpretations.
- Confirm the derived equation by plugging in the original points.

Final Thoughts

The derivation of an implicit plane equation from three points is a foundational technique in geometry, with broad applications across engineering, physics, and computer graphics. Mastering this method enhances spatial reasoning and provides essential tools for solving complex three-dimensional problems.

By following this detailed step-by-step guide, you now have a solid framework to derive the implicit equations of planes passing through any three non-collinear points. Practice with different sets of points to build intuition and proficiency in 3D geometry analysis.

An Implicit Equation For The Plane Passing Through The Points 4 1 3 6 2 8 And 1 1 2 Is

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