Calculate C(5(x2y)i 4(y2 X)j)dr If: (a) C Is The Circle (x7)2 (y1)2=16 Oriented Counterclockwise.

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Understanding how to evaluate line integrals is a fundamental skill in vector calculus, especially when dealing with vector fields around closed curves. In this article, we will explore the detailed process of calculating the line integral

$$
\oint_C \left[5(x^2 y) \mathbf{i} + 4(y^2 x) \mathbf{j} \right] \, dr
$$

where C is the circle defined by $(x - 7)^2 + (y - 1)^2 = 16$, oriented counterclockwise. This problem involves multiple key concepts such as parameterization of circles, vector calculus operators, and properties of line integrals. We will systematically analyze and compute this integral, ensuring a comprehensive understanding of each step.

Understanding the Problem and Key Concepts

Before diving into the calculations, it's essential to understand the components involved:

1. The Vector Field

The integrand is a vector field:

$$
\mathbf{F}(x, y) = 5x^2 y \mathbf{i} + 4 y^2 x \mathbf{j}
$$

This vector field combines polynomial expressions in x and y. Recognizing the structure of such fields helps in applying relevant theorems (like Green's theorem) later.

2. The Curve C

The curve C is described by:

$$
(x - 7)^2 + (y - 1)^2 = 16
$$

which is a circle centered at $(7, 1)$ with a radius of 4. It is oriented counterclockwise, implying a positive orientation.

3. Line Integral over a Closed Curve

The integral:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r}
$$

measures the circulation of the vector field around the curve C.

Mathematical Tools for Evaluation

To compute the line integral efficiently, several methods can be employed:

1. Parameterization of the Curve

Expressing C in parametric form simplifies the integral.

2. Green's Theorem

A powerful theorem relating a line integral around a simple closed curve to a double integral over the region it encloses:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx\,dy
$$

where P and Q are the components of $\mathbf{F}$.

Using Green's theorem often simplifies complex line integrals into manageable double integrals.

3. Computing the Curl or Divergence

In some cases, understanding the curl or divergence of the field can provide insights or simplify calculations.

Step-by-Step Solution Approach

To evaluate the integral, we will proceed through the following systematic steps:

1. Identify the components P and Q of the vector field.
2. Compute the relevant derivatives for Green's theorem.
3. Set up the double integral over the region D bounded by C.
4. Transform the double integral into a coordinate system suitable for the circle (polar coordinates).
5. Calculate the integral, taking into account the orientation.

Step 1: Components of the Vector Field

Given:

$$
\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}
$$

where:

$$
P(x, y) = 5x^2 y
$$

$$
Q(x, y) = 4 y^2 x
$$

Step 2: Applying Green's Theorem

Green's theorem states:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
$$

Calculate the derivatives:

- $\frac{\partial Q}{\partial x}$:

$$
\frac{\partial}{\partial x} (4 y^2 x) = 4 y^2
$$

- $\frac{\partial P}{\partial y}$:

$$
\frac{\partial}{\partial y} (5 x^2 y) = 5 x^2
$$

Therefore, the integrand becomes:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4 y^2 - 5 x^2
$$

Step 3: Setting Up the Double Integral

The integral over the region D:

$$
\iint_D (4 y^2 - 5 x^2) \, dx\,dy
$$

The region D is a circle centered at $(7, 1)$ with radius 4:

$$
(x - 7)^2 + (y - 1)^2 \leq 16
$$

Step 4: Coordinate Transformation to Simplify Integration

To evaluate the double integral, it's convenient to shift the coordinate system to the center of the circle:

Define:
$$
u = x - 7, \quad v = y - 1
$$
then:
$$
x = u + 7, \quad y = v + 1
$$

The Jacobian of the transformation:
$$
\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = 1
$$
since this is a translation, and the Jacobian determinant is 1.

Express the integrand in u, v:

$$
4 y^2 - 5 x^2 = 4 (v + 1)^2 - 5 (u + 7)^2
$$

The domain D in u, v:
$$
u^2 + v^2 \leq 16
$$
which is a circle centered at the origin with radius 4.

Step 5: Evaluating the Integral in u, v Coordinates

The integral becomes:

$$
\iint_{u^2 + v^2 \leq 16} \left[4 (v + 1)^2 - 5 (u + 7)^2 \right] du\,dv
$$

Expanding:

$$
4 (v^2 + 2v + 1) - 5 (u^2 + 14u + 49)
$$

which simplifies to:

$$
4 v^2 + 8 v + 4 - 5 u^2 - 70 u - 245
$$

The integral splits into three parts:

$$
\iint_{D} \left(4 v^2 + 8 v + 4 - 5 u^2 - 70 u - 245 \right) du\,dv
$$

Evaluating the Integral Components

The integral over the disk $u^2 + v^2 \leq 16$ of each term:

1. $\iint_D 4 v^2 \, du\, dv$
2. $\iint_D 8 v \, du\, dv$
3. $\iint_D 4 \, du\, dv$
4. $\iint_D -5 u^2 \, du\, dv$
5. $\iint_D -70 u \, du\, dv$
6. $\iint_D -245 \, du\, dv$

Using symmetry:

- Integrals of odd functions over symmetric regions like u or v centered at zero are zero:
- $\iint_D v \, du\,dv = 0$
- $\iint_D u \, du\, dv = 0$

Remaining:

$$
\boxed{
\int_D 4 v^2 \, du\, dv, \quad
\int_D 4 \, du\, dv, \quad
\int_D -5 u^2 \, du\, dv, \quad
\int_D -245 \, du\, dv
}
$$

Calculating the Area of the Circle D

Area:

$$
A = \pi r^2 = \pi \times 16 = 16 \pi
$$

Integral of Constant Terms

- $\iint_D 4 \, du\, dv = 4 \times A = 4 \times 16 \pi = 64 \pi$
- $\iint_D -245 \, du\, dv = -245$ \

Frequently Asked Questions

What is the main goal when calculating the line integral of the vector field C = 5(x^2 y)i + 4(y^2 x)j along the circle (x/7)^2 + (y/1)^2 = 16?

The main goal is to evaluate the line integral $\int_C (5x^2 y\, i + 4 y^2 x\, j) \cdot dr$ over the given circle, which involves parameterizing the curve and applying line integral techniques.

How do you parametrize the circle (x/7)^2 + (y/1)^2 = 16 for the line integral calculation?

A suitable parametrization is $x = 7\ 4 \cos t$, $y = 1\ 4 \sin t$, with t ranging from 0 to 2π, since the circle has radius 4 and is oriented counterclockwise.

What is the significance of the counterclockwise orientation in evaluating this line integral?

The counterclockwise orientation determines the direction of the parameterization, which affects the sign of the line integral and ensures consistency with positive orientation conventions.

Can Green's theorem be applied to evaluate this line integral, and if so, how?

Yes, Green's theorem can be applied by converting the line integral around the closed curve into a double integral over the region enclosed, simplifying the calculation by evaluating the curl of the vector field.

What is the curl of the vector field C = 5x^2 y i + 4 y^2 x j, and how does it help in calculating the line integral?

The curl in 2D is $\partial(Q)/\partial x - \partial(P)/\partial y$, which helps determine if the vector field is conservative; if the curl is zero, the line integral might be easier to evaluate or reduce to potential functions.

What are the steps to compute the line integral directly using the parametrization?

First, substitute the parametrized $x(t)$ and $y(t)$ into the vector field components, compute dr/dt, then evaluate the integral $\int_0^{2\pi} [F \cdot dr/dt]\, dt$, where F is the vector field.

How does the geometry of the circle influence the calculation of the line integral in this problem?

The circle's radius and orientation determine the limits of integration and the parametrization, affecting the magnitude and sign of the integral, especially when using direct parametrization methods.

What are the common pitfalls to avoid when calculating line integrals over closed curves like this circle?

Common pitfalls include incorrect parametrization, neglecting the orientation, forgetting to account for the differential dr, and misapplying Green's theorem without verifying the conditions.

Additional Resources

Line Integral

Calculating line integrals is a fundamental skill in vector calculus, providing vital insights into fields such as physics, engineering, and mathematics. Today, we'll explore the detailed process of evaluating the line integral of a vector field around a specified curve, focusing on the problem:

$$
\int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{F} = (5x^2 y\,i + 4 y^2 x\,j)
$$

and the curve C is given by the circle:

$$
(x - 7)^2 + (y - 1)^2 = 16
$$

oriented counterclockwise.

This comprehensive guide will break down each step, from understanding the vector field and the curve to applying theorems like Green's theorem for efficient calculation. Whether you're a student brushing up on vector calculus or an engineer applying these calculations practically, this article aims to clarify each concept thoroughly.

Understanding the Components of the Problem

Before diving into calculations, it's essential to dissect the problem's components:

1. The Vector Field $\mathbf{F}$

Given:

$$
\mathbf{F}(x,y) = 5x^2 y\,i + 4 y^2 x\,j
$$

This vector field has two components:

- $P(x, y) = 5x^2 y$ (the i-component)
- $Q(x, y) = 4 y^2 x$ (the j-component)

Understanding the structure of $\mathbf{F}$ helps in applying relevant theorems and choosing the best approach for calculation.

2. The Curve C

The curve:

$$
(x - 7)^2 + (y - 1)^2 = 16
$$

is a circle with center at $(7, 1)$ and radius 4. The orientation is counterclockwise, which is the standard positive orientation in vector calculus.

3. The Goal

Calculate:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r}
$$

which is the line integral of $\mathbf{F}$ along the curve C.

Methodology Overview

Calculating line integrals directly can be complex, especially for curved paths. To simplify the process, vector calculus offers powerful tools:

1. Direct Parametrization of C

- Parameterize C explicitly, then compute the integral.
- Often cumbersome, especially for more complicated fields.

2. Green’s Theorem

- Transforms a line integral around a simple closed curve into a double integral over the area it encloses:

$$
\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
$$

- Particularly efficient for closed curves like circles.

Given that C is a circle, Green's theorem is an ideal approach here.

Applying Green’s Theorem

1. Verify Conditions

- C is a simple, closed, positively oriented curve (counterclockwise).
- $\mathbf{F}$ has continuous partial derivatives in the region.

2. Compute the Partial Derivatives

Given:

$$
P(x, y) = 5x^2 y, \quad Q(x, y) = 4 y^2 x
$$

Calculate:

$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (4 y^2 x) = 4 y^2
$$
$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (5 x^2 y) = 5 x^2
$$

3. Derive the Integrand

Applying Green’s theorem:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
$$

which becomes:

$$
\iint_D (4 y^2 - 5 x^2) dA
$$

Now, the problem reduces to evaluating a double integral over the disk D.

Setting Up the Double Integral

1. Describe the Region D

The disk:

$$
(x - 7)^2 + (y - 1)^2 \leq 16
$$

has center at $(7, 1)$ and radius 4.

2. Coordinate Transformation (Shifted Coordinates)

Since the circle is centered at $(7, 1)$, it's convenient to perform a change of variables:

$$
X = x - 7, \quad Y = y - 1
$$

The region in the (X, Y)-plane is:

$$
X^2 + Y^2 \leq 16
$$

and the integral becomes:

$$
\iint_{X^2 + Y^2 \leq 16} \left[4 (Y + 1)^2 - 5 (X + 7)^2 \right] dX dY
$$

3. Express the Integrand in Terms of X and Y

Expand:

$$
4 (Y + 1)^2 = 4 (Y^2 + 2Y + 1) = 4Y^2 + 8Y + 4
$$
$$
-5 (X + 7)^2 = -5 (X^2 + 14X + 49) = -5X^2 - 70X - 245
$$

Thus, the integrand simplifies to:

$$
(4Y^2 + 8Y + 4) - (5X^2 + 70X + 245) = -5X^2 + 4Y^2 + 8Y - 70X + (4 - 245)
$$

which simplifies further to:

$$
-5X^2 + 4Y^2 + 8Y - 70X - 241
$$

4. Symmetry Considerations

The domain is a circle centered at the origin in (X,Y)-space. When integrating over this symmetric region, terms involving odd powers of X or Y (like X, Y, or XY) will integrate to zero due to symmetry.

In our integrand:

- $-5X^2$ (even in X)
- $4Y^2$ (even in Y)
- $8Y$ (odd in Y)
- $-70X$ (odd in X)
- -241 (constant)

The integrals of the odd terms $8Y$ and $-70X$ over a symmetric circle are zero, leaving:

$$
\iint_{X^2 + Y^2 \leq 16} \left(-5X^2 + 4Y^2 - 241 \right) dX dY
$$

Calculating the Remaining Double Integrals

1. Switch to Polar Coordinates

In the (X,Y)-plane, the circle:

$$
X = r \cos \theta, \quad Y = r \sin \theta, \quad r \in [0, 4], \quad \theta \in [0, 2\pi]
$$

The Jacobian determinant:

$$
dX dY = r dr d\theta
$$

The integrand becomes:

$$
-5 (r \cos \theta)^2 + 4 (r \sin \theta)^2 - 241
$$

which simplifies to:

$$
-5 r^2 \cos^2 \theta + 4 r^2 \sin^2 \theta - 241
$$

Expressed as:

$$
r^2 (-5 \cos^2 \theta + 4 \sin^2 \theta) - 241
$$

Computing the Integrals

```
1. Integrate over \(\theta\)

Note that:

\[
\int_0^{2\pi} \cos^2 \theta d\theta = \pi, \quad \int_0^{2\pi} \sin^2 \theta
d\theta = \pi
\]

and

\[
\int_0^{2\pi} \cos^2 \theta d\theta = \int_0^{2\pi} \sin^2 \theta d\theta =
\pi
\]

Therefore,

\[
\int_0^{2\pi} (-5 \cos^2 \theta + 4 \sin^2 \theta) d\theta = -5 \pi + 4 \pi =
-\pi
\]

2. Integrate over \( r \)

The \( r \)-dependent integral:

\[
\int_0^{
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Calculate C 5 X2y I 4 Y2 X J Dr If A C Is The Circle X7 2 Y1 2 16 Oriented Counterclockwise

Calculate C 5 X2y I 4 Y2 X J Dr If A C Is The Circle X7 2 Y1 2 16 Oriented Counterclockwise

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