Determine If The Vector FieldF=y,x+z2,2yzis Conservative. If It Is, Find A Potential Function.

Determine If The Vector Field F = (y, x + z^2, 2yz) is Conservative. If It Is, Find A Potential Function.

Introduction

Understanding whether a vector field is conservative is a fundamental question in vector calculus, with applications spanning physics, engineering, and mathematics. A conservative vector field is one in which the line integral between any two points is path-independent, meaning the work done by a force field in moving along a path depends only on the endpoints, not the route taken. Such fields are closely associated with potential functions, scalar functions whose gradient yields the vector field in question.

In this article, we will analyze the vector field F = (y, x + z^2, 2yz) to determine whether it is conservative. If it is, we will proceed to find an explicit potential function. The process involves examining properties such as curl, divergence, and the domain of the field, along with solving relevant partial differential equations.

Understanding the Vector Field F

The given vector field is:

$$\mathbf{F}(x, y, z) = (F_1, F_2, F_3) = (y, x + z^2, 2 y z)$$

Each component depends on the variables x, y, and z:

- $F_1 = y$
- $F_2 = x + z^2$
- $F_3 = 2 y z$

Before assessing if F is conservative, it is essential to recall the criteria and properties that characterize conservative vector fields.

Criteria for a Vector Field to be Conservative

A vector field in three-dimensional space is conservative if it satisfies any of the following equivalent conditions:

1. Curl of the Field is Zero

$$\nabla \times \mathbf{F} = \mathbf{0}$$

If the curl of the vector field is zero everywhere in a simply connected domain, then the field is conservative.

2. Existence of a Potential Function

A scalar function $\phi(x, y, z)$ exists such that:
$$\nabla \phi = \mathbf{F}$$
which implies:
$$\frac{\partial \phi}{\partial x} = F_1, \quad$$

\frac{\partial \phi}{\partial y} = F_2, \quad
\frac{\partial \phi}{\partial z} = F_3
\]

3. Path Independence of Line Integrals

If the line integral of \(\mathbf{F}\) between any two points is independent of the path, then \(\mathbf{F}\) is conservative.

Step 1: Compute the Curl of F

The curl is given by:

\[
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}
\]

Calculating each component:

- x-component:

\[
(\nabla \times \mathbf{F})_x = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}
\]

- y-component:

\[
(\nabla \times \mathbf{F})_y = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}
\]

- z-component:

\[
(\nabla \times \mathbf{F})_z = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}
\]

Let's compute each:

Compute \((\nabla \times \mathbf{F})_x\):

\[
\frac{\partial F_3}{\partial y} = \frac{\partial (2 y z)}{\partial y} = 2 z
\]
\[
\frac{\partial F_2}{\partial z} = \frac{\partial (x + z^2)}{\partial z} = 2z
\]
Thus,

\[
(\nabla \times \mathbf{F})_x = 2z - 2z = 0
\]

Compute $(\nabla \times \mathbf{F})_y$:

$$
\frac{\partial F_1}{\partial z} = \frac{\partial y}{\partial z} = 0
$$
$$
\frac{\partial F_3}{\partial x} = \frac{\partial (2 y z)}{\partial x} = 0
$$
Hence,

$$
(\nabla \times \mathbf{F})_y = 0 - 0 = 0
$$

Compute $(\nabla \times \mathbf{F})_z$:

$$
\frac{\partial F_2}{\partial x} = \frac{\partial (x + z^2)}{\partial x} = 1
$$
$$
\frac{\partial F_1}{\partial y} = \frac{\partial y}{\partial y} = 1
$$
So,

$$
(\nabla \times \mathbf{F})_z = 1 - 1 = 0
$$

Result:

$$
\nabla \times \mathbf{F} = (0, 0, 0)
$$

Since the curl of $\mathbf{F}$ is zero everywhere, $\mathbf{F}$ is curl-free.

Step 2: Domain Considerations

For a vector field to be conservative based on curl, the domain must be simply connected. Typically, $\mathbb{R}^3$ (or any domain without holes) satisfies this condition.

Assuming the domain of $\mathbf{F}$ is $\mathbb{R}^3$ or a simply connected subset, the zero curl indicates $\mathbf{F}$ is conservative.

Step 3: Find the Potential Function ϕ

Since $\mathbf{F} = \nabla \phi$, the potential function must satisfy:

$$
\frac{\partial \phi}{\partial x} = y
$$
$$
\frac{\partial \phi}{\partial y} = x + z^2
$$
$$
\frac{\partial \phi}{\partial z} = 2 y z
$$

\]

We will determine ϕ step-by-step by integrating these partial derivatives.

Calculating ϕ from $\partial \phi / \partial x = y$

Integrate with respect to x:

$$
\phi(x, y, z) = \int y \, dx = y x + C_1(y, z)
$$

where $C_1(y, z)$ is an arbitrary function of y, z.

Determine $C_1(y, z)$ using $\partial \phi / \partial y$

Differentiate ϕ:

$$
\frac{\partial \phi}{\partial y} = x + \frac{\partial C_1(y, z)}{\partial y}
$$

Given that:

$$
\frac{\partial \phi}{\partial y} = x + z^2
$$

Compare:

$$
x + \frac{\partial C_1(y, z)}{\partial y} = x + z^2
$$

Subtracting x from both sides:

$$
\frac{\partial C_1(y, z)}{\partial y} = z^2
$$

Integrate with respect to y:

$$
C_1(y, z) = y z^2 + C_2(z)
$$

where $C_2(z)$ is an arbitrary function of z.

Thus, the potential function so far:

$$
\phi(x, y, z) = y x + y z^2 + C_2(z)
$$

Determine $C_2(z)$ using $\partial \phi / \partial z$

Differentiate ϕ:

$$
\frac{\partial \phi}{\partial z} = y \cdot 0 + y \cdot 2 z + \frac{d C_2(z)}{dz} = 2 y z + \frac{d C_2(z)}{dz}
$$

Given that:

$$
\frac{\partial \phi}{\partial z} = 2 y z
$$

Set equal:

$$
2 y z + \frac{d C_2(z)}{dz} = 2 y z
$$

Subtract $2 y z$:

$$
\frac{d C_2(z)}{dz} = 0
$$

Integrate:

$$
C_2(z) = \text{constant} = C
$$

Final potential function:

\[
\boxed{
\phi(x, y, z) = x y +

Frequently Asked Questions

What is the first step to determine if the vector field $F = (y, x + z^2, 2yz)$ is conservative?

The first step is to verify whether the vector field is defined on a simply connected domain and check if its curl is zero.

How do you compute the curl of the vector field $F = (y, x + z^2, 2yz)$?

You compute the curl by taking the cross product of the del operator with F, i.e., $\nabla \times F$, calculating the partial derivatives accordingly.

What does it indicate if the curl of $F = (y, x + z^2, 2yz)$ is zero everywhere?

If the curl is zero everywhere and the domain is simply connected, then the vector field is conservative.

How can we verify if the domain of the vector field is simply connected?

Typically, the domain is R^3 or a subset without holes; if the domain is all of R^3 or similar, it is simply connected. If there are obstacles or holes, it may not be simply connected.

What is the process to find a potential function for a conservative vector field?

To find the potential function ϕ, integrate the components of F with respect to their respective variables, ensuring the derivatives match F's components.

Can you demonstrate how to find the potential function for $F = (y, x + z^2, 2yz)$?

Yes. For example, integrate y with respect to x (which gives $\phi = xy + g(y,z)$), then differentiate this with respect to y and set equal to the second component to find g(y,z), and similarly proceed for the third component.

What is the potential function ϕ for the vector field $F = (y, x + z^2, 2yz)$?

The potential function is $\phi(x, y, z) = xy + yz^2 + C$, where C is a constant.

Why is it important to verify both the curl and the domain when determining if a vector field is conservative?

Because a zero curl guarantees a conservative field only if the domain is simply connected; otherwise, the field may not be conservative despite a zero curl.

Additional Resources

Determine If The Vector Field F = y i + (x + z^2) j + 2 y z k is Conservative is a fundamental problem in vector calculus, especially when analyzing the nature of vector fields and their potential functions. Whether a vector field is conservative has significant implications in physics and engineering, such as understanding whether a force field is derivable from a potential energy function. This article provides a comprehensive exploration of the criteria used to determine if a vector field is conservative, along with detailed steps to find a potential function if it exists.

Understanding Conservative Vector Fields

What Is a Conservative Vector Field?

A vector field F in three-dimensional space is said to be conservative if there exists a scalar potential function $\phi(x, y, z)$ such that:

$$\mathbf{F} = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

This means that F can be expressed as the gradient of some scalar function, implying that the work done by the field in moving a particle between two points is path-independent.

Key features of conservative fields:
- Path independence of line integrals.
- Zero curl in simply connected regions.
- Existence of potential functions.

Applications:
- Gravitational and electrostatic fields.
- Force fields derived from potential energy.

Given Vector Field F

The vector field provided is:

$$
\mathbf{F} = y\, \mathbf{i} + (x + z^2)\, \mathbf{j} + 2 y z\, \mathbf{k}
$$

Expressed component-wise:

$$
F_x = y,\quad F_y = x + z^2,\quad F_z = 2 y z
$$

The goal is to determine whether F is conservative, and if so, to find a potential function $\phi(x, y, z)$ such that:

$$
\nabla \phi = \mathbf{F}
$$

which implies:

$$
\frac{\partial \phi}{\partial x} = y,\quad \frac{\partial \phi}{\partial y} = x + z^2,\quad \frac{\partial \phi}{\partial z} = 2 y z
$$

Step 1: Check the Curl of the Vector Field

One of the primary criteria for a vector field to be conservative in a simply connected domain is that its curl must be zero:

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

Calculate the curl:

\[

$$
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
F_x & F_y & F_z
\end{vmatrix}
$$

which expands to:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i}
- \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \mathbf{j}
+ \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}
$$

Calculate each component:

- $\frac{\partial F_z}{\partial y} = \frac{\partial (2 y z)}{\partial y} = 2 z$
- $\frac{\partial F_y}{\partial z} = \frac{\partial (x + z^2)}{\partial z} = 2 z$
- $\frac{\partial F_z}{\partial x} = \frac{\partial (2 y z)}{\partial x} = 0$
- $\frac{\partial F_x}{\partial z} = \frac{\partial y}{\partial z} = 0$
- $\frac{\partial F_y}{\partial x} = \frac{\partial (x + z^2)}{\partial x} = 1$
- $\frac{\partial F_x}{\partial y} = \frac{\partial y}{\partial y} = 1$

Now, compute each curl component:

- i-component:

$$
2 z - 2 z = 0
$$

- j-component:

$$
- (0 - 0) = 0
$$

- k-component:

$$
1 - 1 = 0
$$

Hence,

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

Since the curl is zero everywhere and the domain (all of $\mathbb{R}^3$) is simply connected, F is conservative.

Step 2: Find the Potential Function φ(x, y, z)

Given that F is conservative, we can find φ by integrating the components.

Method:

- Integrate $F_x = y$ with respect to x:

$$
\phi(x, y, z) = \int y\, dx = y x + h(y,z)
$$

where $h(y,z)$ is an arbitrary function of y and z (since partial differentiation with respect to x treats y, z as constants).

- Differentiate φ with respect to y:

$$
\frac{\partial \phi}{\partial y} = x + \frac{\partial h(y,z)}{\partial y}
$$

But from the given vector field:

$$
\frac{\partial \phi}{\partial y} = F_y = x + z^2
$$

Set equal:

$$
x + \frac{\partial h(y,z)}{\partial y} = x + z^2
$$

which implies:

$$
\frac{\partial h(y,z)}{\partial y} = z^2
$$

- Integrate with respect to y:

$$
h(y,z) = y z^2 + g(z)
$$

where $g(z)$ is an arbitrary function of z.

- Now, differentiate φ with respect to z:

$$
\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}(x y + y z^2 + g(z)) = y \cdot 0 + y \cdot 2 z + g'(z) = 2 y z + g'(z)
$$

From the vector field:

$$
\frac{\partial \phi}{\partial z} = F_z = 2 y z
$$

\]

Set equal:

$$
2 y z + g'(z) = 2 y z
$$

which implies:

$$
g'(z) = 0
$$

Thus, $g(z)$ is a constant (say, C), which can be ignored in the potential function as it adds a constant.

Final Potential Function

Combining all parts, the potential function is:

$$
\boxed{
\phi(x, y, z) = x y + y z^2 + C
}
$$

where C is an arbitrary constant.

Summary and Conclusion

- The vector field $\mathbf{F} = y\, \mathbf{i} + (x + z^2)\, \mathbf{j} + 2 y z\, \mathbf{k}$ is conservative because its curl is zero across its domain.
- The potential function for $\mathbf{F}$ is:

$$
\boxed{
\phi(x, y, z) = x y + y z^2 + C
}
$$

- This potential function satisfies $\nabla \phi = \mathbf{F}$, confirming the field's conservative nature.

Features and Implications

Pros of Recognizing Conservative Fields:
- Simplifies calculations of work and energy in physics.

- Ensures path-independent line integrals.
- Facilitates solving differential equations involving the field.

Cons or Limitations:
- The region must be simply connected; otherwise, curl zero does not guarantee conservativeness.
- Not all vector fields with zero curl are globally conservative if the domain isn't simply connected.

Additional notes:
- The method demonstrated here, integrating components stepwise, is standard but requires careful checking of compatibility conditions.
- When a vector field isn't conservative, alternative methods such as vector potential or numerical approaches are necessary.

In conclusion, by verifying the curl condition and integrating component-wise, we established that the vector field \(\mathbf{F}\) is conservative and successfully derived its potential function. This process underscores the importance of fundamental vector calculus tools in analyzing physical and mathematical systems.

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