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Evaluating surface integrals is a fundamental aspect of vector calculus, allowing us to compute the flux of a vector field across a surface or the surface area weighted by a scalar function. In this article, we focus on the specific surface integral of the form $\iint_S Z^2 \, dS$, where S is a portion of the paraboloid $X = Y^2 + Z^2$ bounded between $X=0$ and $X=1$. This problem involves understanding the geometry of the surface, parametrization, and calculating the differential surface element dS. Our goal is to evaluate the integral step-by-step, providing a comprehensive explanation suitable for students and practitioners of multivariable calculus.

Understanding the Surface S and the Integral

Surface Description

The surface S is given by the paraboloid $X = Y^2 + Z^2$. The problem specifies the part of this paraboloid between $X=0$ and $X=1$. Geometrically, this is a "slice" of the paraboloid originating at the vertex (where $X=0$) and extending outward until $X=1$.

Key points about the surface S:

- It is a paraboloid opening along the X-axis.
- The bounds for X are from 0 to 1.
- The cross-sections at a fixed X are circles with radius $\sqrt{X}$ in the YZ-plane.

What is Being Integrated?

The integrand is Z^2, a scalar function of the surface. The goal is to integrate this function over the surface S, considering the surface element dS. The integral is:

$$
\iint_S Z^2 \, dS
$$

To evaluate this, we need an explicit parametrization of S, the expression for dS, and the limits of integration.

Parametrization of the Surface S

Choosing a Suitable Parameterization

Since the surface is given by $X = Y^2 + Z^2$, a natural approach is to use $\mathbf{r}(Y,Z)$ as a parametrization:

$$
\mathbf{r}(Y,Z) = (Y^2 + Z^2,\, Y,\, Z)
$$

Here, Y and Z are free parameters. However, to incorporate the bounds on X, which vary from 0 to 1, we need to express the limits of Y and Z in terms of X, or vice versa.

Expressing Y and Z in terms of X

Since $X = Y^2 + Z^2$, for a fixed X, the cross-section is the circle $Y^2 + Z^2 = X$. Therefore, for each X, the points on the surface satisfy:

$$
Y^2 + Z^2 = X,\quad 0 \leq X \leq 1
$$

In the (Y,Z)-plane, the cross-section at a fixed X is a circle of radius $\sqrt{X}$. To parametrize this circle, we introduce an angle parameter θ:

$$
\begin{cases}
Y = \sqrt{X} \cos \theta \\
Z = \sqrt{X} \sin \theta
\end{cases}
$$

with $\theta \in [0, 2\pi]$. The parameter X varies from 0 to 1, and θ varies from 0 to 2π.

Complete Parametrization

Combining these, the parametrization of the surface becomes:

$$
\boxed{
\mathbf{r}(X, \theta) = (X,\, \sqrt{X} \cos \theta,\, \sqrt{X} \sin \theta)
}
$$

with the limits:

$$
X \in [0,1],\quad \theta \in [0, 2\pi]
$$

Calculating the Surface Element dS

Partial Derivatives of $\mathbf{r}(X,\theta)$

To find dS, we compute the vectors $\mathbf{r}_X$ and $\mathbf{r}_\theta$:

$$
\mathbf{r}_X = \frac{\partial \mathbf{r}}{\partial X} = \left(1,\, \frac{1}{2\sqrt{X}}\cos \theta,\, \frac{1}{2\sqrt{X}}\sin \theta \right)
$$

$$
\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \left(0,\,-\sqrt{X}\sin \theta,\, \sqrt{X}\cos \theta \right)
$$

Cross Product to Find dS

The magnitude of the cross product $\mathbf{r}_X \times \mathbf{r}_\theta$ gives the differential surface element:

$$
dS = |\mathbf{r}_X \times \mathbf{r}_\theta|\, dX\, d\theta
$$

Calculating the cross product:

\[
\mathbf{r}_X \times \mathbf{r}_\theta =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & \frac{1}{2\sqrt{X}}\cos \theta & \frac{1}{2\sqrt{X}}\sin \theta \\
0 & -\sqrt{X}\sin \theta & \sqrt{X}\cos \theta
\end{vmatrix}
\]

Compute each component:

- \(i\)-component:

\[
\left(\frac{1}{2\sqrt{X}}\cos \theta \times \sqrt{X}\cos \theta \right) - \left(\frac{1}{2\sqrt{X}}\sin \theta \times (-\sqrt{X}\sin \theta) \right)
\]

Simplify:

\[
\frac{1}{2\sqrt{X}} \times \sqrt{X} \cos^2 \theta + \frac{1}{2\sqrt{X}} \times \sqrt{X} \sin^2 \theta = \frac{1}{2} \cos^2 \theta + \frac{1}{2} \sin^2 \theta = \frac{1}{2}
\]

- \(j\)-component:

\[
- \left(1 \times \sqrt{X} \cos \theta - \frac{1}{2\sqrt{X}}\sin \theta \times 0 \right) = - \left(\sqrt{X} \cos \theta - 0 \right) = - \sqrt{X} \cos \theta
\]

- \(k\)-component:

\[
1 \times (-\sqrt{X} \sin \theta) - \frac{1}{2\sqrt{X}}\cos \theta \times 0 = - \sqrt{X} \sin \theta
\]

Thus,

\[
\mathbf{r}_X \times \mathbf{r}_\theta = \left(\frac{1}{2},\, - \sqrt{X} \cos \theta,\, - \sqrt{X} \sin \theta \right)
\]

The magnitude:

\[
|\mathbf{r}_X \times \mathbf{r}_\theta| = \sqrt{ \left(\frac{1}{2}\right)^2 + (\sqrt{X} \cos \theta)^2 + (\sqrt{X} \sin \theta)^2 }
\]

$$
= \sqrt{ \frac{1}{4} + X \cos^2 \theta + X \sin^2 \theta } = \sqrt{ \frac{1}{4} + X (\cos^2 \theta + \sin^2 \theta) } = \sqrt{ \frac{1}{4} + X }
$$

since $\cos^2 \theta + \sin^2 \theta = 1$.

Expression for dS

$$
dS = |\mathbf{r}_X \times \mathbf{r}_\theta|\, dX\, d\theta = \sqrt{X + \frac{1}{4}}\, dX\, d\theta
$$