Find Equations Of Both Lines Through The Point (2, 3) That Are Tangent To The Parabola Y = X2 + X

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When exploring the fascinating world of calculus and analytic geometry, a common problem involves finding lines that are tangent to a given curve and pass through a specific point. In this article, we will delve into a detailed process to find both lines passing through the point (2, 3) that are tangent to the parabola y = x2 + x. This problem combines concepts of derivatives, equations of lines, and tangent conditions, providing a comprehensive example suitable for students and enthusiasts aiming to strengthen their understanding of tangency and line equations.

Understanding the Problem

The task is to determine the equations of two lines that meet the following criteria:

- Both lines pass through the point (2, 3).
- Each line is tangent to the parabola y = x2 + x.

Graphically, this means that each line touches the parabola at exactly one point, and that point of contact lies somewhere on the parabola. Since both lines go through (2, 3), they intersect at that point, but only touch the parabola at a single point each, where they share a common tangent.

This problem involves a combination of algebraic and calculus techniques:

- Equation of a line passing through (2, 3): The general form y = m(x - 2) + 3, where m is the slope.
- Condition for tangency: The line intersects the parabola at exactly one point, implying the quadratic equation obtained by substituting the line into the parabola has a single (repeated) root.

Step-by-Step Solution Approach

To find the tangent lines, we follow a systematic approach:

1. Write the general equation of the lines passing through (2, 3).
2. Substitute the line equation into the parabola to find the intersection points.
3. Apply the tangency condition (discriminant zero) to find the slopes (m).
4. Determine the equations of the tangent lines using the slopes found.

Let's explore each step in detail.

Step 1: Equation of a line passing through (2, 3)

Any line passing through the point (2, 3) can be written as:

$$
y = m(x - 2) + 3
$$

where m is the slope of the line.

Step 2: Substituting the line into the parabola

Given the parabola:

$$
y = x^2 + x
$$

and the line:

$$
y = m(x - 2) + 3
$$

To find the points of intersection, set these equal:

$$
x^2 + x = m(x - 2) + 3
$$

Rearranged as:

$$
x^2 + x - m(x - 2) - 3 = 0
$$

Expanding the right side:

$$
x^2 + x - m x + 2 m - 3 = 0
$$

Bring all to one side:

$$
x^2 + x - m x + 2 m - 3 = 0
$$

Combine like terms:

$$
x^2 + (1 - m) x + (2 m - 3) = 0
$$

This quadratic in x represents the intersection points between the parabola and the line.

Step 3: Applying the tangency condition

For the line to be tangent to the parabola, this quadratic must have exactly one solution (a repeated root). This occurs when the discriminant (D) of the quadratic is zero:

$$
D = [(1 - m)]^2 - 4 \times 1 \times (2 m - 3) = 0
$$

Calculate the discriminant:

$$
(1 - m)^2 - 4 (2 m - 3) = 0
$$

Expand:

$$
(1 - 2 m + m^2) - 8 m + 12 = 0
$$

Simplify:

$$
m^2 - 2 m + 1 - 8 m + 12 = 0
$$

Combine like terms:

$$
m^2 - 10 m + 13 = 0
$$

This quadratic in m gives the slopes of the tangent lines.

Step 4: Solving for the slopes m

Solve the quadratic:

$$
m^2 - 10 m + 13 = 0
$$

Use the quadratic formula:

$$
m = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 13}}{2}
$$

Calculate discriminant:

$$
100 - 52 = 48
$$

Thus,

$$
m = \frac{10 \pm \sqrt{48}}{2}
$$

Simplify $\sqrt{48}$:

$$
\sqrt{48} = \sqrt{16 \times 3} = 4 \sqrt{3}
$$

So,

$$
m = \frac{10 \pm 4 \sqrt{3}}{2} = 5 \pm 2 \sqrt{3}
$$

Therefore, the two slopes are:

$$
\boxed{
m_1 = 5 + 2 \sqrt{3}
}
$$

\[
\boxed{
m_2 = 5 - 2 \sqrt{3}
\]

Equations of the Tangent Lines

Now that we have the slopes, we can write the equations of the tangent lines passing through (2, 3).

Recall the line form:

\[
y = m(x - 2) + 3
\]

For \(m_1 = 5 + 2 \sqrt{3}\):

\[
\boxed{
\text{Line 1: } y = (5 + 2 \sqrt{3})(x - 2) + 3
}
\]

For \(m_2 = 5 - 2 \sqrt{3}\):

\[
\boxed{
\text{Line 2: } y = (5 - 2 \sqrt{3})(x - 2) + 3
}
\]

These are the equations of the two lines passing through (2, 3) that are tangent to the parabola y = x2 + x.

Verification of the Tangency

To ensure correctness, it’s prudent to verify that these lines are indeed tangent to the parabola.

Method:

- Substitute the line equation into the parabola.
- Confirm the quadratic in x has a discriminant of zero.

Let's verify for one of the lines.

Verification for Line 1:

Line 1:

$$
y = (5 + 2 \sqrt{3})(x - 2) + 3
$$

Express as:

$$
y = (5 + 2 \sqrt{3}) x - 2 (5 + 2 \sqrt{3}) + 3
$$

Calculate the constant term:

$$
-2 (5 + 2 \sqrt{3}) + 3 = -10 - 4 \sqrt{3} + 3 = -7 - 4 \sqrt{3}
$$

So,

$$
y = (5 + 2 \sqrt{3}) x - 7 - 4 \sqrt{3}
$$

Now, set this equal to the parabola:

$$
x^2 + x = (5 + 2 \sqrt{3}) x - 7 - 4 \sqrt{3}
$$

Bring all to one side:

$$
x^2 + x - (5 + 2 \sqrt{3}) x + 7 + 4 \sqrt{3} = 0
$$

Combine like terms:

$$
x^2 + [1 - (5 + 2 \sqrt{3})] x + 7 + 4 \sqrt{3} = 0
$$

Simplify the coefficient of x:

\[
1 - 5 - 2 \sqrt{3} = -4 - 2 \sqrt{3}
\]

Therefore, quadratic:

\[
x^2 + (-4 - 2 \sqrt{3}) x + 7 + 4 \sqrt{3} = 0
\]

Calculate the discriminant D:

\[
D = (-4 - 2 \sqrt{3})^2 - 4 \times 1 \times (7 + 4 \sqrt{3})
\]

Compute:

\[
(-4)^2 + 2 \times -4 \times -2 \sqrt{3} + (-2 \sqrt{3})^2 - 4(7 + 4 \sqrt{3})
\]

But more straightforwardly:

\[
(-4 - 2 \sqrt{3})^2 = (-4)^2 + 2 \times -4 \times -2 \sqrt{3} + (-2 \sqrt{3})^2
\]

Calculate each term:

- \((-4)^2 =

Frequently Asked Questions

What are the equations of the tangent lines to the parabola $y = x^2 + x$ passing through the point (2, 3)?

The tangent lines passing through (2, 3) have equations $y = m(x - 2) + 3$, where m satisfies the condition that these lines are tangent to $y = x^2 + x$. Solving yields two possible slopes, and the corresponding tangent line equations can be found accordingly.

How do you determine the slopes of the tangent lines from point (2, 3) to the parabola $y = x^2 + x$?

You set up the condition that the line passing through (2, 3) with slope m intersects the parabola $y = x^2 + x$ at exactly one point, leading to a quadratic in x. Solving the discriminant condition (discriminant zero) gives the slopes m of the tangent lines.

What is the method to find the equations of tangent lines through a given point to a parabola?

The method involves: 1) writing the equation of a line through the point with variable slope m; 2) substituting into the parabola's equation; 3) setting the discriminant of resulting quadratic to zero to ensure tangency; 4) solving for m, then for the line equations.

Can you provide the explicit equations of the tangent lines from (2, 3) to $y = x^2 + x$?

Yes. Calculations show the slopes are $m = -1$ and $m = 2$. The tangent lines are: $y = -1(x - 2) + 3$, which simplifies to $y = -x + 5$; and $y = 2(x - 2) + 3$, which simplifies to $y = 2x - 1$.

Why do the tangent lines from a point outside a parabola often come in pairs?

Because a parabola is a conic section, a point outside it typically has two tangent lines passing through it—each touching the parabola at exactly one point—corresponding to the two real solutions of the tangent condition.

Additional Resources

Tangency: Unlocking the Geometric Elegance of Lines and Parabolas

In the realm of coordinate geometry, the interplay between lines and curves offers a fascinating glimpse into the harmony of mathematics. Among the myriad of problems, finding tangent lines to a parabola through a specific point is both a classic challenge and an elegant demonstration of algebraic and geometric principles. Today, we'll delve into a comprehensive exploration of how to determine the equations of both lines passing through the point (2, 3) that are tangent to the parabola $y = x^2 + x$. This journey combines algebraic mastery with geometric intuition, providing insights valuable to students, educators, and enthusiasts alike.

Understanding the Core Concepts

Before we jump into the calculations, it's essential to ground ourselves in the fundamental ideas that underpin this problem:

- Tangent Line: A line that touches a curve at exactly one point without crossing it at that point. The point of contact is called the point of tangency.

- Parabola $y = x^2 + x$: A quadratic curve opening upwards, shifted and stretched in the coordinate plane, characterized by a specific algebraic form.

- Point (2, 3): The fixed point through which both tangent lines must pass.

The challenge is to find the equations of all lines passing through (2, 3) that are tangent to the parabola $y = x^2 + x$.

Step 1: Setting Up the General Equation of the Line

To proceed systematically:

- Let the line passing through (2, 3) have an unknown slope m.

- The equation of such a line can be written in point-slope form:

$$
y - 3 = m (x - 2)
$$

- Simplify to slope-intercept form:

$$
y = m (x - 2) + 3
$$

$$
y = m x - 2 m + 3
$$

Our goal: determine the values of m such that this line is tangent to the parabola $y = x^2 + x$.

Step 2: Formulating the Tangency Condition

The line and the parabola are tangent if they intersect at exactly one point—that is, the quadratic equation formed by equating their y-values has a single solution.

Set the parabola equal to the line:

$$
x^2 + x = m x - 2 m + 3
$$

Rearranged as:

$$
x^2 + x - m x + 2 m - 3 = 0
$$

Combine like terms:

$$
x^2 + (1 - m) x + (2 m - 3) = 0
$$

This quadratic in x:

$$
x^2 + (1 - m) x + (2 m - 3) = 0
$$

has exactly one solution for x when the discriminant D is zero:

$$
D = [(1 - m)]^2 - 4 \times 1 \times (2 m - 3) = 0
$$

Calculating the discriminant:

$$
D = (1 - m)^2 - 4 (2 m - 3)
$$
$$
D = (1 - 2 m + m^2) - 8 m + 12
$$
$$
D = m^2 - 2 m + 1 - 8 m + 12
$$
$$
D = m^2 - 10 m + 13
$$

Set D = 0 for tangency:

$$
m^2 - 10 m + 13 = 0
$$

Step 3: Solving for the Slopes m

The quadratic in m:

\[
m^2 - 10 m + 13 = 0
\]

Applying the quadratic formula:

\[
m = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 13}}{2}
\]

Calculate the discriminant:

\[
\Delta_m = 100 - 52 = 48
\]

Therefore,

\[
m = \frac{10 \pm \sqrt{48}}{2}
\]

Simplify \(\sqrt{48}\):

\[
\sqrt{48} = \sqrt{16 \times 3} = 4 \sqrt{3}
\]

Thus,

\[
m = \frac{10 \pm 4 \sqrt{3}}{2}
\]

Divide numerator and denominator:

\[
m = 5 \pm 2 \sqrt{3}
\]

These are the slopes of the two tangent lines passing through (2, 3).

Step 4: Writing the Equations of the Tangent Lines

Recall the general form of the lines:

\[

y = m x - 2 m + 3
\]

Substitute each value of m:

For \(m = 5 + 2 \sqrt{3} \):

\[
y = (5 + 2 \sqrt{3}) x - 2 (5 + 2 \sqrt{3}) + 3
\]

Simplify:

\[
y = (5 + 2 \sqrt{3}) x - 10 - 4 \sqrt{3} + 3
\]

\[
y = (5 + 2 \sqrt{3}) x - 7 - 4 \sqrt{3}
\]

For \(m = 5 - 2 \sqrt{3} \):

\[
y = (5 - 2 \sqrt{3}) x - 2 (5 - 2 \sqrt{3}) + 3
\]

Simplify:

\[
y = (5 - 2 \sqrt{3}) x - 10 + 4 \sqrt{3} + 3
\]

\[
y = (5 - 2 \sqrt{3}) x - 7 + 4 \sqrt{3}
\]

Final Equations of the Tangent Lines

Thus, the two lines passing through (2, 3) that are tangent to the parabola \(y = x^2 + x \) are:

1. Line 1:

\[
\boxed{
\quad y = (5 + 2 \sqrt{3}) x - 7 - 4 \sqrt{3}
}

\]

2. Line 2:

\[
\boxed{
\quad y = (5 - 2 \sqrt{3}) x - 7 + 4 \sqrt{3}
}
\]

Geometric and Analytical Insights

This problem exemplifies a beautiful intersection of algebra and geometry. By imposing the condition for tangency—namely, that the quadratic in \(x\) has a discriminant of zero—we effectively translate a geometric condition into an algebraic one. The resulting slopes are irrational, involving \(\sqrt{3}\), which underscores the richness of solutions in the real plane.

Key takeaways include:

- The significance of the discriminant in determining tangency conditions.
- The method of substituting the line equation into the parabola to derive a quadratic.
- The utility of quadratic formulas in solving for slopes.
- The importance of checking the discriminant to confirm tangency (single intersection point).

Extensions and Further Exploration

While this article has focused specifically on the parabola \(y = x^2 + x \) and a fixed point (2, 3), the methodology extends to a variety of other curves and points:

- Different curves: For example, circles, ellipses, or hyperbolas.
- Different points: Finding tangent lines through other points.
- Multiple tangents: In cases where more than two tangent lines exist, the process involves solving higher-degree equations.

For advanced students, consider exploring how these principles adapt when the parabola's equation changes or when the point of tangency is fixed and the line varies.

Conclusion

Finding the equations of lines passing through a given point and tangent to a parabola exemplifies the elegance and power of algebraic methods in geometry. Through systematic derivation—starting from the general line equation, setting up the tangency condition via the discriminant, solving quadratic equations for slopes, and then writing the explicit line equations—we've uncovered the precise lines that satisfy these conditions.

This problem not only reinforces foundational concepts but also showcases how algebra and geometry interplay to produce clear, exact solutions. Whether you're a student aiming to master quadratic equations, a teacher illustrating the beauty of conic sections, or an enthusiast appreciating mathematical symmetry, this exploration offers a compelling example of the depth and elegance inherent in coordinate geometry.

Happy exploring the curves and lines that shape our mathematical universe!

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