Find The Arc Length Of The Plane Curve Given By The Parametric Equations: X=14t^2 ,y=3t^2 14,0t3

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Understanding how to find the arc length of a plane curve described by parametric equations is a fundamental concept in calculus, particularly in the study of curves and their properties. This article provides a comprehensive step-by-step guide to calculating the arc length for the specific parametric equations: X=14t^2 and y=3t^2 + 14, with the parameter t ranging from 0 to 3. Whether you're a student preparing for exams or a mathematics enthusiast seeking to deepen your understanding, this detailed explanation will clarify the process and methodology involved.

Introduction to Arc Length in Parametric Form

Arc length represents the total distance traveled along a curve between two points. When dealing with parametric equations, where the coordinates of a point on the curve are expressed as functions of a parameter t, the calculation involves integrating the differential element of the curve's length over the specified interval.

General formula for the arc length (L):

$$
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

Where:

- $x = x(t)$

- $y = y(t)$

- t varies from a to b

Understanding the Given Parametric Equations

The specific equations provided are:

$$
x(t) = 14 t^2
$$

$$
y(t) = 3 t^2 + 14
$$

with the interval:

$$
0 \leq t \leq 3
$$

Let's analyze these equations.

Nature of the Curve

- Both x and y are quadratic functions of t.

- The x-coordinate depends on t^2, scaled by 14.

- The y-coordinate depends on t^2, scaled by 3, with an added constant 14.

This indicates a parabolic shape, and as t varies from 0 to 3, the curve traces out a specific segment of this parabola.

Step-by-Step Calculation of the Arc Length

The calculation involves several steps:

1. Find the derivatives dx/dt and dy/dt.
2. Compute the integrand $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.
3. Set up the definite integral over the interval $t = 0$ to $t = 3$.
4. Evaluate the integral to find the total arc length.

1. Derivatives of $x(t)$ and $y(t)$

Calculate dx/dt:

$$
\frac{dx}{dt} = \frac{d}{dt} (14 t^2) = 28 t
$$

Calculate dy/dt:

$$
\frac{dy}{dt} = \frac{d}{dt} (3 t^2 + 14) = 6 t
$$

2. Expression for the integrand

Substitute derivatives into the arc length formula:

$$
\sqrt{(28 t)^2 + (6 t)^2} = \sqrt{784 t^2 + 36 t^2} = \sqrt{820 t^2}
$$

Simplify:

$$
\sqrt{820 t^2} = \sqrt{820} \, |t|
$$

Since t ranges from 0 to 3, and $t \geq 0$, $|t| = t$. Therefore:

$$
\text{Integrand} = \sqrt{820} \, t
$$

Note that $\sqrt{820}$ can be simplified further if desired, but for the purpose of calculation, keeping it as $\sqrt{820}$ suffices.

3. Setting up the definite integral

The arc length L is:

$$
L = \int_{0}^{3} \sqrt{820} \, t \, dt

\]

Pulling out the constant:

$$
L = \sqrt{820} \int_{0}^{3} t \, dt
$$

4. Computing the integral

Evaluate:

$$
\int_{0}^{3} t \, dt = \left[\frac{t^2}{2} \right]_0^3 = \frac{3^2}{2} - 0 = \frac{9}{2}
$$

Thus, the total arc length:

$$
L = \sqrt{820} \times \frac{9}{2}
$$

Expressed as a simplified radical:

$$
\sqrt{820} = \sqrt{4 \times 205} = 2 \sqrt{205}
$$

Therefore:

$$
L = 2 \sqrt{205} \times \frac{9}{2} = 9 \sqrt{205}
$$

Final Result:

$$
\boxed{
L = 9 \sqrt{205}
}
$$

This is the exact length of the curve between $t=0$ and $t=3$.

Additional Insights and Applications

Understanding how to compute the arc length is crucial for various applications, such as:

- Designing curves in engineering and architecture.
- Calculating the length of roller coaster tracks or roads.
- Analyzing the geometry of physical systems.

Additional notes:

- The process involves derivatives and integrals, fundamental in calculus.
- Simplifying radicals makes the result easier to interpret.
- For numerical approximation, compute $\sqrt{205}$ approximately, then multiply by 9.

Numerical Approximation

$$
\sqrt{205} \approx 14.32
$$

Thus:

$$
L \approx 9 \times 14.32 \approx 128.88
$$

The approximate length of the curve over the interval is about 128.88 units.

Summary of Steps for Finding Arc Length of Parametric Curves

To summarize, the general approach involves:

1. Differentiate $x(t)$ and $y(t)$ with respect to t.
2. Compute the integrand $\sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}$.
3. Set up the definite integral from a to b.
4. Perform the integration, simplifying where possible.
5. Express the result in radical form or decimal approximation.

By mastering these steps, you can find the arc length of any parametric curve within a specified interval, expanding your calculus toolkit for analyzing curves in geometry, physics, and engineering.

Conclusion

Calculating the arc length of the plane curve given by the parametric equations $x=14 t^2$ and $y=3 t^2 + 14$ over the interval $0 \leq t \leq 3$ results in an exact length of $9 \sqrt{205}$, approximately 128.88 units. This process illustrates the power of calculus in understanding the geometry of curves, transforming complex shapes into manageable integrals. Whether for academic purposes or practical applications, mastering this technique is essential for analyzing the physical and mathematical properties of curves.

Keywords: arc length, parametric equations, calculus, curve length, integral, derivatives, plane curve, mathematics, geometry, calculus tutorial

Frequently Asked Questions

How do I find the arc length of a plane curve given parametric equations?

To find the arc length of a plane curve given parametric equations x(t) and y(t), you use the formula: L = $\int$ from a to b of $\sqrt{}$[(dx/dt)^2 + (dy/dt)^2] dt. First, compute the derivatives dx/dt and dy/dt, then integrate their squares' square root over the specified t-interval.

Given $x=14t^2$ and $y=3t^2$, how do I set up the arc length integral?

First, find dx/dt = 28t and dy/dt = 6t. The arc length L from t=a to t=b is then L = □□□ □[$(28t)^2 + (6t)^2$] dt = □□□ □($784t^2 + 36t^2$) dt = □□□ □($820t^2$) dt. Simplify to L = □□□ □820 |t| dt.

What are the limits of integration for t in this problem?

The problem specifies t from 0 to 3, so the limits of integration are t=0 and t=3.

How do I evaluate the integral for the arc length in this case?

Since t is from 0 to 3 and t □ 0, □820 |t| simplifies to □820 t. The integral becomes L = □820 □□^{3} t dt = □820 [$t^2/2$] from 0 to 3 = □820 (9/2) = (9/2) □820.

What is the approximate numeric value of the arc length?

Calculate □820 □ 28.62. Then, L □ (9/2) 28.62 □ 4.5 28.62 □ 128.79 units.

Can the arc length formula be simplified further for these parametric equations?

Yes. Since both derivatives involve t and the integral reduces to a straightforward polynomial, the calculation simplifies to L = (9/2) □820, as shown. For other curves, similar steps apply, involving derivatives and integration.

Additional Resources

Find The Arc Length Of The Plane Curve Given By The Parametric Equations: X=14t^2 ,y=3t^2 14,0t3

Understanding the elegance of mathematical curves often leads us into the realm of calculus, where the concepts of derivatives and integrals reveal the hidden properties of these geometric entities. Today, we delve into a specific problem: calculating the arc length of a plane curve defined parametrically by the equations $X=14t^2$ and $y=3t^2$. This problem not only demonstrates fundamental calculus principles but also showcases how parametric equations serve as powerful tools to describe complex curves beyond simple functions. Whether you're a student brushing up on calculus or an enthusiast exploring the geometry of curves, this comprehensive guide will walk you through the process with clarity and depth.

Understanding the Problem: What Is Arc Length?

Before diving into calculations, it's crucial to grasp what arc length signifies in the context of a curve.

What Is Arc Length?

Arc length refers to the distance measured along a curve between two points. Imagine tracing your finger along a winding path; the length of that path is the arc length. Unlike straight-line distance, which measures the shortest path between two points, arc length accounts for the actual route along the curve, capturing its bends and twists.

Why Is Calculating Arc Length Important?

Determining the arc length of a curve has numerous applications:
- Engineering: Designing roads, ramps, or tracks that follow specific paths.
- Physics: Calculating the distance traveled along a trajectory.
- Computer Graphics: Rendering smooth curves and animations.
- Mathematics: Analyzing the properties of curves, such as their length and curvature.

Parametric Equations: The Foundation of the Curve

The curve in question is described parametrically by two equations involving a parameter t:

$$
X(t) = 14t^2
$$

$$
Y(t) = 3t^2
$$

The parameter t varies over a specified interval, which, in this case, is from $t=0$ to $t=3$.

Interpreting the Equations

- The X-coordinate depends quadratically on t with a coefficient of 14.
- The y-coordinate also depends quadratically but with a coefficient of 3.
- Both depend on t^2, implying the curve exhibits symmetry and is smoothly increasing as t increases.

Why Use Parametric Equations?

Parametric equations are particularly useful when:

- The curve cannot be easily expressed as a single function $y=f(x)$.
- The curve involves complex shapes or motions.
- The relationship between x and y is more naturally described through a parameter.

Deriving the Arc Length Formula for Parametric Curves

Calculating the arc length for a parametric curve involves an integral that accounts for the infinitesimal distances along the curve.

The General Formula

Given parametric equations $x(t)$ and $y(t)$, the arc length L from $t=a$ to $t=b$ is:

$$
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

This formula sums up the infinitesimal distances ds along the curve, where:

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

Applying to Our Curve

For our specific equations:

$$
x(t) = 14t^2
$$

$$
y(t) = 3t^2
$$

we need to compute dx/dt and dy/dt, then integrate the square root of their squares over the interval $t=0$ to $t=3$.

Calculating Derivatives: The First Step

The derivatives of $x(t)$ and $y(t)$ with respect to t are:

$$
\frac{dx}{dt} = \frac{d}{dt} (14t^2) = 28t
$$

$$
\frac{dy}{dt} = \frac{d}{dt} (3t^2) = 6t
$$

These derivatives give the rates at which the x and y coordinates change as t varies.

Formulating the Integral for Arc Length

Substituting the derivatives into the arc length formula:

$$
L = \int_{0}^{3} \sqrt{(28t)^2 + (6t)^2} \, dt
$$

Simplify inside the square root:

$$
L = \int_{0}^{3} \sqrt{784t^2 + 36t^2} \, dt
$$

$$
L = \int_{0}^{3} \sqrt{820t^2} \, dt
$$

Since $t \geq 0$ in the interval [0,3], we can write:

$$
L = \int_{0}^{3} \sqrt{820} \, t \, dt
$$

The square root of 820 can be simplified:

$$
\sqrt{820} = \sqrt{4 \times 205} = 2 \sqrt{205}
$$

Thus, the integral becomes:

$$
L = \int_{0}^{3} 2 \sqrt{205} \, t \, dt
$$

Note: The integrand simplifies to a linear function in t, making the integral straightforward.

Performing the Integration

The integral simplifies to:

$$
L = 2 \sqrt{205} \int_{0}^{3} t \, dt
$$

Calculating the integral:

$$
\int t \, dt = \frac{t^2}{2}
$$

Applying the limits:

$$
\int_{0}^{3} t \, dt = \frac{3^2}{2} - 0 = \frac{9}{2}
$$

\]

Substituting back:

$$
L = 2 \sqrt{205} \times \frac{9}{2} = \sqrt{205} \times 9
$$

Therefore, the exact arc length is:

$$
L = 9 \sqrt{205}
$$

Interpreting the Result: Numerical Approximation

To gain a tangible sense of this length, approximate $\sqrt{205}$:

$$
\sqrt{205} \approx 14.3178
$$

Multiplying:

$$
L \approx 9 \times 14.3178 \approx 128.86
$$

Hence, the arc length of the curve from $t=0$ to $t=3$ is approximately 128.86 units.

Conclusion: Significance and Applications of the Calculation

Calculating the arc length of a parametric curve like \(X=14t^2 \) and \(y=3t^2 \) exemplifies the power of calculus in understanding geometric properties. The process involved:

- Deriving the derivatives of the parametric equations.
- Applying the arc length integral formula.
- Simplifying the integral to find an exact expression.
- Approximating the result for practical understanding.

This approach is widely applicable across various fields:

- Engineering: Designing curves with precise lengths for manufacturing or construction.
- Physics: Computing the actual distance traveled along a path described parametrically.
- Mathematics: Analyzing properties of complex curves that cannot be easily expressed as functions of \(x \) or \(y \).

By mastering these techniques, students and professionals can unlock deeper insights into the geometry of curves, enabling accurate modeling and analysis of real-world phenomena. Whether dealing with the winding roads of a racetrack or the intricate paths of particles in motion, understanding how to find the arc length of a parametric curve is an essential skill in the mathematician's toolkit.

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