Find The Area Of The Surface. The Part Of The Plane 5x + 2y + Z = 10 That Lies In The First Octant

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Calculating the surface area of a plane segment in the first octant is a fundamental problem in multivariable calculus. It combines understanding the geometric representation of planes, the concept of surface integrals, and the application of mathematical techniques to find the area of a specific portion of a surface. In this comprehensive guide, we will explore the process of finding the surface area of the part of the plane $5x + 2y + z = 10$ that lies within the first octant, providing detailed steps, explanations, and examples to facilitate a clear understanding of the procedure.

Understanding the Problem

Before delving into calculations, it's essential to understand the problem's components:

What is the Plane Equation?

- The given plane is described by the equation $5x + 2y + z = 10$.
- This is a linear equation in three variables, representing a flat surface in three-dimensional space.

What Does "Lies in the First Octant" Mean?

- The first octant is the region where $x \geq 0$, $y \geq 0$, and $z \geq 0$.
- The portion of the plane that lies in this octant is bounded by the coordinate planes and the plane itself.

Objective

- To compute the surface area of the part of the plane that exists within the first octant.

Mathematical Foundations

Calculating the surface area involves understanding surface integrals and parameterization of surfaces:

Surface Area Formula for a Surface z = f(x, y)

- When a surface is defined as $z = f(x, y)$, the surface area over a region R in the xy-plane is given by:

$$\text{Surface Area} = \iint_{R} \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \, dx\,dy$$

- This formula accounts for the surface's slope in both directions, providing the total area.

Parameterization of the Surface

- Alternatively, the surface can be parameterized as (x, y, z), where z is expressed as a function of x and y.

Step-by-Step Solution Approach

The process involves several stages:

1. Express the Surface as a Function z = f(x, y)

- From the plane equation $5x + 2y + z = 10$, solve for z:

$z = 10 - 5x - 2y$

- This representation is valid in the region where $z \ge 0$, which corresponds to the part of the plane in the first octant.

2. Determine the Region R in the xy-Plane

- The bounds of x and y are determined by the conditions:
- $x \ge 0$
- $y \ge 0$
- $z \ge 0 \rightarrow 10 - 5x - 2y \ge 0$

- From $z \ge 0$:

$10 - 5x - 2y \ge 0$

- Rearranged as:

$5x + 2y \le 10$

- Since x and y are non-negative, the region R is bounded by:
- $x = 0$
- $y = 0$
- $5x + 2y = 10$

- The region R is thus a triangle with vertices at:
- (0, 0)
- (2, 0) (when y=0, x=2)
- (0, 5) (when x=0, y=5)

3. Set Up the Double Integral

- The surface area S is:

$S = \iint_{R} \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \, dx\,dy$

- Compute the partial derivatives:

$\frac{\partial z}{\partial x} = -5$

$\frac{\partial z}{\partial y} = -2$

- Plug into the formula:

$S = \iint_{R} \sqrt{1 + (-5)^2 + (-2)^2} \, dx\,dy$

$S = \iint_{R} \sqrt{1 + 25 + 4} \, dx\,dy = \iint_{R} \sqrt{30} \, dx\,dy$

- Since $\sqrt{30}$ is constant, the integral simplifies to:

$S = \sqrt{30} \times \text{Area of region R}$

Calculating the Area of Region R

The region R is a triangle in the xy-plane with vertices at (0,0), (2,0), and (0,5). Its area can be found using the formula for the area of a triangle given coordinates or by integrating over the bounds:

Method 1: Using Coordinates

- The area of triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3):

$\text{Area} = \frac{1}{2} | x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) |$

- Substitute points:

(0,0), (2,0), (0,5):

$\text{Area} = \frac{1}{2} | 0(0 - 5) + 2(5 - 0) + 0(0 - 0) |$

$= \frac{1}{2} | 0 + 10 + 0 | = \frac{1}{2} \times 10 = 5$

Method 2: Using Integration

- Set up the limits:

- For y:

y varies from 0 to $y = \frac{10 - 5x}{2}$

- For x:

x varies from 0 to 2

- The area:

$\text{Area} = \int_{x=0}^{2} \int_{y=0}^{(10 - 5x)/2} dy\, dx$

- Compute the inner integral:

$\int_{0}^{(10 - 5x)/2} dy = \frac{10 - 5x}{2}$

- Outer integral:

$\int_{0}^{2} \frac{10 - 5x}{2} dx$

$= \frac{1}{2} \int_{0}^{2} (10 - 5x) dx$

$= \frac{1}{2} \left[10x - \frac{5}{2} x^2 \right]_0^2$

$= \frac{1}{2} \left[10 \times 2 - \frac{5}{2} \times 4 \right]$

$= \frac{1}{2} \left[20 - 10 \right] = \frac{1}{2} \times 10 = 5$

- Both methods agree that the area of R is 5.

Final Calculation of Surface Area

- Recall:

$S = \sqrt{30} \times \text{Area of R}$

- Substituting the area:

$S = \sqrt{30} \times 5$

- Simplify:

$S = 5 \sqrt{30}$

- Approximate numerical value:

$\sqrt{30} \approx 5.4772$

$S \approx 5 \times 5.4772 \approx 27.386$

Summary of the Results

- The surface area of the part of the plane 5x + 2y + z = 10 that lies in the first octant is exactly:

$\boxed{5 \sqrt{30}}$ square units

- The approximate numerical value is around 27.386 square units.

Additional Insights and Applications

Understanding how to compute the surface area of a plane segment is vital in various fields, including engineering, physics, and computer graphics. The techniques demonstrated here can be extended to more complex surfaces and regions, aiding in tasks such as:

- Surface integrals in physics for calculating flux
- Material science for surface area estimation
- Computer graphics for rendering 3D objects

Key Takeaways

- Express the surface as a function $z = f(x, y)$.
-

Frequently Asked Questions

How do you determine the surface area of the part of the plane $5x + 2y + z = 10$ lying in the first octant?

To find the surface area, first express z in terms of x and y ($z = 10 - 5x - 2y$), identify the region in the xy-plane bounded by the intercepts, and then compute the surface area using the surface area formula for a graph $z = f(x,y)$.

What are the intercepts of the plane $5x + 2y + z = 10$ in the first octant?

The intercepts are at $(x, y, z) = (2, 0, 0)$, $(0, 5, 0)$, and $(0, 0, 10)$, which define the triangular region in the xy-plane over which the surface area must be integrated.

What is the formula for calculating the surface area of a surface $z = f(x,y)$ over a region R?

The surface area A is given by the double integral $A = \iint_R \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dx\, dy$, where R is the projection of the surface onto the xy-plane.

How do you set up the double integral for the surface area in this problem?

The double integral is set up over the triangular region with vertices at (0,0), (2,0), and (0,5), with the integrand $\sqrt{1 + 25 + 4} = \sqrt{30}$, and limits for y from 0 to (5/2)x, and x from 0 to 2.

What is the value of the surface area for the part of the plane in the first octant?

The surface area is $(25/2)\sqrt{30}$ square units.

Why is it important to consider the first octant when finding the surface area of the plane?

Because the problem specifies the part of the plane in the first octant, which restricts x, y, and z to non-negative values, ensuring the integration region is correctly bounded and the surface area is accurately calculated over that specific portion.

Additional Resources

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Introduction

Calculating the surface area of a plane segment in three-dimensional space is a fundamental problem in multivariable calculus, with applications spanning physics, engineering, and computer graphics. This task involves understanding how the plane intersects with a specific region—in this case, the first octant—and then systematically deriving the surface area of that portion.

This detailed exploration will guide you through the step-by-step process of finding the surface area of the part of the plane $5x + 2y + z = 10$ that exists within the first octant, where $x \geq 0$,

$y \geq 0$, and $z \geq 0$.

Understanding the Problem

Before diving into calculations, it's essential to clarify what the problem entails:

- The plane is given by the equation $5x + 2y + z = 10$.
- The region of interest is the part of this plane that lies in the first octant, the portion where all coordinate variables are non-negative.
- The goal is to find the surface area of this specific segment.

Geometric Interpretation of the Plane in the First Octant

1. Intercepts of the Plane

To define the bounded region on the plane, identify where the plane intersects the coordinate axes:

- x-intercept: Set $y = 0$, $z = 0$:

$$
5x + 0 + 0 = 10 \Rightarrow x = 2
$$

- y-intercept: Set $x = 0$, $z = 0$:

$$
0 + 2y + 0 = 10 \Rightarrow y = 5
$$

- z-intercept: Set $x = 0$, $y = 0$:

$$
0 + 0 + z = 10 \Rightarrow z = 10
$$

Since the region is in the first octant, the bounds are constrained by these intercepts:

- $0 \leq x \leq 2$
- $0 \leq y \leq 5$

- $0 \leq z \leq 10$

However, the surface segment is only the part of the plane within these bounds, which form a triangular region on the plane.

Visualizing the Surface Region

The intersection of the plane with the coordinate planes results in a triangular region with vertices at:

- $A = (0, 0, 10)$ (z-intercept)
- $B = (2, 0, 0)$ (x-intercept)
- $C = (0, 5, 0)$ (y-intercept)

This triangle lies on the plane and is bounded by the lines between these points.

Step 1: Expressing the Surface

The surface is given by the equation:

$$
z = 10 - 5x - 2y
$$

for the region where $z \geq 0$, which implies:

$$
10 - 5x - 2y \geq 0
$$
$$
5x + 2y \leq 10
$$

and, of course, with $x, y \geq 0$.

Step 2: Setting up the Surface Area Integral

The surface area element for a graph $z = f(x, y)$ over a region R in the xy-plane is:

$$
dS = \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \, dx \, dy
$$

- Find the partial derivatives:

$$
\frac{\partial z}{\partial x} = -5
$$

$$
\frac{\partial z}{\partial y} = -2
$$

- Compute the integrand:

$$
\sqrt{1 + (-5)^2 + (-2)^2} = \sqrt{1 + 25 + 4} = \sqrt{30}
$$

Thus,

$$
dS = \sqrt{30} \, dx \, dy
$$

The total surface area S is:

$$
S = \iint_{R} \sqrt{30} \, dx \, dy
$$

where R is the triangular region in the xy-plane bounded by:

$$
\begin{cases}
x \geq 0 \\
y \geq 0 \\
5x + 2y \leq 10
\end{cases}
$$

Step 3: Describing the Region R

To compute the double integral, define R explicitly:

- For a fixed x, the maximum y satisfying $5x + 2y \leq 10$ is:

$$
2y \leq 10 - 5x \Rightarrow y \leq \frac{10 - 5x}{2}
$$

- The bounds for y:

$$
0 \leq y \leq \frac{10 - 5x}{2}
$$

- The bounds for x:

$$
0 \leq x \leq 2
$$

Because at $x=0$, $y \leq 5$, which matches the y-intercept, and at $x=2$, $y=0$.

Step 4: Computing the Integral

The surface area becomes:

$$
S = \sqrt{30} \int_{x=0}^{2} \int_{y=0}^{\frac{10 - 5x}{2}} dy \, dx
$$

First, integrate with respect to y:

$$
S = \sqrt{30} \int_{0}^{2} \left[y \right]_0^{\frac{10 - 5x}{2}} dx = \sqrt{30} \int_{0}^{2} \frac{10 - 5x}{2} dx
$$

Simplify the integrand:

$$
S = \frac{\sqrt{30}}{2} \int_{0}^{2} (10 - 5x) dx
$$

Now, perform the integral:

$$
\int (10 - 5x) dx = 10x - \frac{5x^2}{2}
$$

Evaluate from 0 to 2:

$$
\left[10x - \frac{5x^2}{2} \right]_0^{2} = (10 \times 2) - \frac{5 \times 2^2}{2} = 20 - \frac{5 \times 4}{2} = 20 - \frac{20}{2} = 20 - 10 = 10
$$

Therefore,

$$
S = \frac{\sqrt{30}}{2} \times 10 = 5 \sqrt{30}
$$

Final Result:

$$
\boxed{
\text{Surface Area} = 5 \sqrt{30}
}
$$

This is the exact surface area of the part of the plane $5x + 2y + z = 10$ lying within the first octant.

Additional Insights and Considerations

1. Geometric Significance of the Result

- The calculation confirms that the surface area depends on the geometry of the bounded region.
- The factor $\sqrt{30}$ arises from the gradients of the plane, indicating how steeply the surface rises.
- The region itself is a right triangle with vertices at the intercepts, making the problem manageable.

2. Alternative Methods

- Using parametric representation: Instead of expressing z explicitly, parametrize the surface with parameters x and y.
- Projection method: Project the surface onto the xy-plane and integrate as we did, which simplifies computations.

3. Extensions and Applications

- Similar methods are applicable to other planes, curved surfaces, or more complex geometries.
- Surface area calculations are crucial in physics for flux computations, in engineering for material estimations, and in computer graphics for rendering.

Summary

To synthesize the entire process:

- Identify intercepts to determine the bounded region.
- Express the surface as a function $z = f(x, y)$.
- Calculate the partial derivatives to find the surface element dS.
- Set up the double integral over the region R in the xy-plane.
- Evaluate the integral to find the total surface area.

By following these systematic steps, the surface area of the specified plane segment in the first octant is precisely $\boxed{5 \sqrt{30}}$.

Final Remarks

Understanding how to find the surface area of a plane segment within a specified boundary is a vital skill in multivariable calculus. The approach outlined here emphasizes

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