Find The Orthogonal Projection Of 9e1 Onto The Subspace Of R4 Spanned By 2 2 1 0 And -2 2 0 1

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Understanding the concept of orthogonal projection in vector spaces is fundamental in linear algebra, especially when working with subspaces in R4. In this article, we will thoroughly explore how to find the orthogonal projection of the vector 9e1 onto the subspace of R4 spanned by the vectors (2, 2, 1, 0) and (-2, 2, 0, 1). This process involves concepts such as basis vectors, orthogonality, and projection formulas, which are crucial for applications across various fields, including computer graphics, data analysis, and engineering. Through detailed steps, explanations, and illustrative examples, we aim to make this topic accessible and comprehensive.

Understanding the Problem

Before diving into the calculation, it’s essential to clarify the components of the problem:

What is 9e1?

- 9e1 refers to the vector in R4 with 9 in the first component and zeros elsewhere:
$$
9e_1 = (9, 0, 0, 0)
$$
- The goal is to project this vector onto a subspace spanned by two vectors.

The Subspace of R4

- The subspace is spanned by the vectors:
$$
v_1 = (2, 2, 1, 0)
$$
$$
v_2 = (-2, 2, 0, 1)
$$
- These vectors form a basis (assuming linear independence), defining a 2-dimensional subspace within R4.

Mathematical Foundations

A solid grasp of the underlying mathematics is necessary to perform the projection accurately.

Orthogonal Projection in R4

- The orthogonal projection of a vector u onto a subspace W spanned by basis vectors $\{v_1, v_2, ..., v_k\}$ can be expressed as:

$$
\text{proj}_W(\mathbf{u}) = \sum_{i=1}^k \alpha_i v_i
$$

where the coefficients α_i are determined to minimize the distance $\|\mathbf{u} - \text{proj}_W(\mathbf{u})\|$.

Projection Formula Using Gram-Schmidt

- To find the coefficients α_i, especially when the basis vectors are not orthogonal, we can use the Gram-Schmidt process to orthogonalize the basis vectors, then project accordingly.

Alternative: Projection Matrix Method

- Alternatively, if we assemble the basis vectors into a matrix A, the projection can be computed using:

$$
\text{proj}_W(\mathbf{u}) = A (A^T A)^{-1} A^T \mathbf{u}
$$

- This method simplifies calculations, especially with computational tools.

Step-by-Step Solution

Let's now systematically proceed with the calculation.

Step 1: Define the basis vectors

- Given:

$$
v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}
$$

$$
v_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}
$$

- The vector to project:

$$
u = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Step 2: Assemble the basis into a matrix A

- Form matrix A with basis vectors as columns:
$$
A = \begin{bmatrix}
2 & -2 \\
2 & 2 \\
1 & 0 \\
0 & 1
\end{bmatrix}
$$

Step 3: Compute $A^T A$

- Calculate the Gram matrix:
$$
A^T A = \begin{bmatrix}
v_1 \cdot v_1 & v_1 \cdot v_2 \\
v_2 \cdot v_1 & v_2 \cdot v_2
\end{bmatrix}
$$
- Compute each dot product:
- $v_1 \cdot v_1 = 2^2 + 2^2 + 1^2 + 0^2 = 4 + 4 + 1 + 0 = 9$
- $v_1 \cdot v_2 = (2)(-2) + (2)(2) + (1)(0) + (0)(1) = -4 + 4 + 0 + 0 = 0$
- $v_2 \cdot v_2 = (-2)^2 + 2^2 + 0^2 + 1^2 = 4 + 4 + 0 + 1 = 9$

- Thus,
$$
A^T A = \begin{bmatrix}
9 & 0 \\
0 & 9
\end{bmatrix}
$$

Step 4: Compute $A^T u$

- Calculate:
$$
A^T u = \begin{bmatrix}
v_1 \cdot u \\
v_2 \cdot u
\end{bmatrix}
$$
- Dot products:

- \(v_1 \cdot u = 29 + 20 + 10 + 00 = 18 + 0 + 0 + 0 = 18\)
- \(v_2 \cdot u = -29 + 20 + 00 + 10 = -18 + 0 + 0 + 0 = -18\)

- So,
\[
A^T u = \begin{bmatrix} 18 \\ -18 \end{bmatrix}
\]

Step 5: Solve for \(\alpha\) using the projection formula

- Since \(A^T A\) is diagonal, invert it easily:
\[
(A^T A)^{-1} = \begin{bmatrix}
1/9 & 0 \\
0 & 1/9
\end{bmatrix}
\]
- Compute \(\alpha = (A^T A)^{-1} A^T u\):
\[
\alpha = \begin{bmatrix}
1/9 & 0 \\
0 & 1/9
\end{bmatrix}
\begin{bmatrix} 18 \\ -18 \end{bmatrix}
= \begin{bmatrix}
18/9 \\ -18/9
\end{bmatrix}
= \begin{bmatrix}
2 \\ -2
\end{bmatrix}
\]

Step 6: Compute the orthogonal projection

- The projection vector is:
\[
\text{proj}_W(u) = A \alpha
\]
- Multiply:
\[
\begin{bmatrix}
2 & -2 \\
2 & 2 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\ -2

\end{bmatrix}
=
\begin{bmatrix}
22 + (-2)(-2) \\
22 + 2(-2) \\
12 + 0(-2) \\
02 + 1(-2)
\end{bmatrix}
=
\begin{bmatrix}
4 + 4 \\
4 - 4 \\
2 + 0 \\
0 - 2
\end{bmatrix}
=
\begin{bmatrix}
8 \\ 0 \\ 2 \\ -2
\end{bmatrix}
\]

Final Result and Interpretation

The orthogonal projection of the vector \(9e_1 = (9, 0, 0, 0)\) onto the subspace spanned by \(v_1\) and \(v_2\) is:

\[
\boxed{
\text{proj}_W(9e_1) = (8, 0, 2, -2)
}
\]

This vector represents the closest point in the subspace to \(9e_1\), measured in Euclidean distance. It lies within the span of the basis vectors and is orthogonal to the difference

Frequently Asked Questions

What is the orthogonal projection of the vector 9e1 onto the subspace of R4 spanned by vectors (2, 2, 1, 0) and (-2, 2, 0, 1)?

The orthogonal projection of 9e1 onto the subspace is the sum of its projections onto each spanning vector. First, project onto each vector separately and then sum these projections to find the total projection.

How do I compute the orthogonal projection of a vector onto a subspace spanned by multiple vectors?

To compute the orthogonal projection of a vector onto a subspace spanned by vectors v1, v2, ..., vk, you can set up and solve the system using the Gram-Schmidt process or project onto each basis vector individually and sum the results, ensuring the basis vectors are orthogonal or orthonormal.

What is the significance of orthogonal projection in linear algebra?

Orthogonal projection helps find the closest vector in a subspace to a given vector, which is useful in least squares problems, data approximation, and solving systems with minimal error.

How do I find the projection of 9e1 onto the vector (2, 2, 1, 0)?

Use the formula: projection = (v · u / u · u) u, where v is 9e1 and u is the vector (2, 2, 1, 0). Compute the dot products, then multiply by u.

How do I find the projection of 9e1 onto the vector (-2, 2, 0, 1)?

Apply the same projection formula: (v · u / u · u) u, with v = 9e1 and u = (-2, 2, 0, 1). Calculate the dot products and multiply accordingly.

What is the step-by-step process to find the orthogonal projection of a vector onto a subspace spanned by two vectors?

1. Find projections of the vector onto each spanning vector individually. 2. Sum these projections to get the total projection. 3. Optionally, orthogonalize the spanning vectors using Gram-Schmidt for more precise calculations.

Can the projection of 9e1 be expressed as a linear combination of the spanning vectors?

Yes, the projection can be written as a linear combination of the spanning vectors (2, 2, 1, 0) and (-2, 2, 0, 1), with coefficients determined by the projection formulas.

What is the actual numerical value of the orthogonal

projection of 9e1 onto the given subspace?

Calculating step-by-step: first project onto each vector, then sum the results. The projection onto (2, 2, 1, 0) yields (18/9)(2, 2, 1, 0) = 2(2, 2, 1, 0) = (4, 4, 2, 0). The projection onto (-2, 2, 0, 1) yields (18/9)(-2, 2, 0, 1) = 2(-2, 2, 0, 1) = (-4, 4, 0, 2). Summing these gives (0, 8, 2, 2).

What is the final projection vector of 9e1 onto the subspace?

The final orthogonal projection of 9e1 onto the subspace spanned by (2, 2, 1, 0) and (-2, 2, 0, 1) is (0, 8, 2, 2).

Additional Resources

Find The Orthogonal Projection Of 9e1 Onto The Subspace Of R4 Spanned By 2 2 1 0 And -2 2 0 1

In the realm of linear algebra, understanding how vectors relate to subspaces is fundamental. One particularly important concept is the orthogonal projection, which allows us to find the closest point in a subspace to a given vector. This process has applications across various fields, including computer graphics, data science, and engineering. Today, we'll explore a specific example: determining the orthogonal projection of the vector $9e_1$ onto a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = (2, 2, 1, 0)$ and $\mathbf{v}_2 = (-2, 2, 0, 1)$. This problem exemplifies the practical steps involved in projection calculations, emphasizing both the theoretical framework and the computational techniques.

Understanding the Problem: The Geometric and Algebraic Context

Before diving into the calculations, it’s crucial to understand what the problem entails. The goal is to find the orthogonal projection of a specific vector onto a subspace. Let’s break down the components:

- The Vector to Project: $\mathbf{u} = 9e_1 = (9, 0, 0, 0)$

- The Subspace W: Spanned by vectors $\mathbf{v}_1 = (2, 2, 1, 0)$ and $\mathbf{v}_2 = (-2, 2, 0, 1)$

The main question is: What is the point $\mathbf{p} \in W$ such that the vector $\mathbf{u} - \mathbf{p}$ is orthogonal to every vector in W?

Mathematically, the orthogonal projection of $\mathbf{u}$ onto W can be expressed as:

$$
\text{proj}_W(\mathbf{u}) = \mathbf{p} = a \mathbf{v}_1 + b \mathbf{v}_2
$$

where $a, b \in \mathbb{R}$. The task reduces to finding a and b such that:

$$
\mathbf{u} - \mathbf{p} \perp W
$$

which means:

$$
(\mathbf{u} - \mathbf{p}) \cdot \mathbf{v}_1 = 0 \quad \text{and} \quad
(\mathbf{u} - \mathbf{p}) \cdot \mathbf{v}_2 = 0
$$

Mathematical Foundations: Orthogonal Projections in $\mathbb{R}^n$

The process of orthogonal projection relies on the principles of inner products and subspace orthogonality. Here's a quick overview:

- Inner Product: In $\mathbb{R}^4$, the standard inner product of vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ is:

$$
\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4
$$

- Orthogonal Complement: The difference $\mathbf{u} - \mathbf{p}$ is orthogonal to W, meaning it's orthogonal to each basis vector $\mathbf{v}_1$ and $\mathbf{v}_2$.

- Projection: The orthogonal projection of $\mathbf{u}$ onto W is the vector $\mathbf{p} \in W$ such that the residual $\mathbf{u} - \mathbf{p}$ is orthogonal to W.

To find $\mathbf{p}$, one can set up a system of equations based on the orthogonality conditions.

Formulating the System: Setting Up Equations for a and b

Given the basis vectors and the vector $\mathbf{u}$, the projection $\mathbf{p}$ can be written as:

$$
\mathbf{p} = a \mathbf{v}_1 + b \mathbf{v}_2
$$

The orthogonality conditions are:

$$
(\mathbf{u} - \mathbf{p}) \cdot \mathbf{v}_1 = 0
$$
$$
(\mathbf{u} - \mathbf{p}) \cdot \mathbf{v}_2 = 0
$$

Substituting $\mathbf{p}$:

$$
(\mathbf{u} - a \mathbf{v}_1 - b \mathbf{v}_2) \cdot \mathbf{v}_1 = 0
$$
$$
(\mathbf{u} - a \mathbf{v}_1 - b \mathbf{v}_2) \cdot \mathbf{v}_2 = 0
$$

which simplifies to the linear system:

$$
\begin{cases}
\mathbf{u} \cdot \mathbf{v}_1 = a (\mathbf{v}_1 \cdot \mathbf{v}_1) + b (\mathbf{v}_2 \cdot \mathbf{v}_1) \\
\mathbf{u} \cdot \mathbf{v}_2 = a (\mathbf{v}_1 \cdot \mathbf{v}_2) + b (\mathbf{v}_2 \cdot \mathbf{v}_2)
\end{cases}
$$

Calculating each inner product:

- $\mathbf{u} \cdot \mathbf{v}_1 = 9 \times 2 + 0 + 0 + 0 = 18$
- $\mathbf{u} \cdot \mathbf{v}_2 = 9 \times (-2) + 0 + 0 + 0 = -18$
- $\mathbf{v}_1 \cdot \mathbf{v}_1 = 2^2 + 2^2 + 1^2 + 0 = 4 + 4 + 1 + 0 = 9$
- \(\mathbf{v}_2 \cdot \mathbf{v}_2 = (-2)^2 + 2^2 + 0 + 1^2 = 4 + 4 + 0 + 1

$= 9$
- $\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 \times (-2) + 2 \times 2 + 1 \times 0 + 0 \times 1 = -4 + 4 + 0 + 0 = 0$

Thus, the system reduces to:

$$
\begin{cases}
18 = 9a + 0b \\
-18 = 0a + 9b
\end{cases}
$$

which simplifies further:

$$
\begin{cases}
9a = 18 \Rightarrow a = 2 \\
9b = -18 \Rightarrow b = -2
\end{cases}
$$

Computing the Projection Vector

With the coefficients a and b determined, the projection $\mathbf{p}$ is:

$$
\mathbf{p} = a \mathbf{v}_1 + b \mathbf{v}_2 = 2 \times (2, 2, 1, 0) + (-2) \times (-2, 2, 0, 1)
$$

Calculating each component:

- First component:

$$
2 \times 2 + (-2) \times (-2) = 4 + 4 = 8
$$

- Second component:

$$
2 \times 2 + (-2) \times 2 = 4 - 4 = 0
$$

- Third component:

\[
2 \times 1 + (-2) \times 0 = 2 + 0 = 2
\]

- Fourth component:

\[
2 \times 0 + (-2) \times 1 = 0 - 2 = -2
\]

Therefore, the orthogonal projection of \(9e_1 \) onto the subspace \(W \) is:

\[
\boxed{
\text{proj}_W(9e_1) = (8, 0, 2, -2)
}
\]

Interpreting the Result and Its Significance

This projection vector \((8, 0, 2, -2) \) is the closest point within the subspace \(W \) to \(\mathbf{u} = (9, 0, 0, 0) \). Geometrically, this means that among all vectors spanned by \(\mathbf{v}_1 \) and \(\mathbf{v}_

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