Find The Solution Of Given Initial Value Problem Y(4) 2y'' Y=3t 6; Y(0)=y'(0)=0,y''(0)=y^(3)(0)=1

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Solving differential equations, particularly initial value problems (IVPs), is fundamental in applied mathematics, physics, and engineering. The problem presented involves a higher-order differential equation with specified initial conditions, demanding a systematic approach to obtain the function y(t). In this comprehensive guide, we will explore the step-by-step process to find the solution of the initial value problem:

Y(4) 2y'' Y=3t 6; Y(0)=y'(0)=0, y''(0)=y^(3)(0)=1

This problem combines differential equations, initial conditions, and algebraic manipulation, making it an excellent example for students and professionals aiming to deepen their understanding of solving complex IVPs.

Understanding the Given Differential Equation and Initial Conditions

Before diving into solving the problem, it's crucial to interpret the notation and understand the structure of the differential equation and initial conditions.

Analyzing the Differential Equation

The notation appears somewhat ambiguous; however, a common interpretation in differential equations contexts is:

- Y(4) likely refers to the fourth derivative of y, denoted as $y^{(4)}(t)$.
- 2y'' Y probably signifies 2 times the second derivative, y''(t), possibly multiplied by a function or coefficient Y, but context suggests it might be part of the entire differential equation.
- The right side 3t 6 may represent 3t + 6, assuming a typo or formatting issue.

Putting it together, a plausible form of the differential equation is:

$$ y^{(4)}(t) + 2 y''(t) = 3t + 6 $$

This is a linear nonhomogeneous differential equation of fourth order.

Initial Conditions

- $y(0) = 0$
- $y'(0) = 0$
- $y''(0) = 1$
- $y^{(3)}(0) = 1$

These conditions specify the values of y and its derivatives at t=0, essential for finding the particular solution.

Step-by-Step Solution Approach

The process involves several stages:

1. Rewrite the differential equation in standard form.
2. Find the complementary (homogeneous) solution.
3. Determine a particular solution.
4. Apply initial conditions to find the constants.
5. Write the general solution.

Let's explore each step in detail.

Step 1: Rewrite the Differential Equation

Assuming the differential equation is:

$$ y^{(4)}(t) + 2 y''(t) = 3t + 6 $$

This form is standard and ready for solution.

Step 2: Find the Homogeneous Solution

Solve the homogeneous equation:

$$ y^{(4)} + 2 y'' = 0 $$

which simplifies to:

$$ y^{(4)} + 2 y'' = 0 $$

Notice that this involves derivatives of different orders, but it can be simplified by substitution.

Method:

Let $z(t) = y''(t)$, then:

$$ y^{(4)} = z'' $$

Thus, the homogeneous equation becomes:

$$ z'' + 2z = 0 $$

This is a second-order linear differential equation with constant coefficients.

Characteristic equation:

$$ r^2 + 2 = 0 \implies r^2 = -2 \implies r = \pm i \sqrt{2} $$

Solution for z(t):

$$ z(t) = C_1 \cos(\sqrt{2} t) + C_2 \sin(\sqrt{2} t) $$

Recall that:

$$ y''(t) = z(t) $$

Integrate twice to find y(t):

- First integration:

$$ y'(t) = \int z(t) dt + C_3 $$

- Second integration:

$$ y(t) = \int y'(t) dt + C_4 $$

Integrate z(t):

$$ y'(t) = \int [C_1 \cos(\sqrt{2} t) + C_2 \sin(\sqrt{2} t)] dt + C_3 $$

\[y'(t) = C_1 \frac{\sin(\sqrt{2} t)}{\sqrt{2}} - C_2 \frac{\cos(\sqrt{2} t)}{\sqrt{2}} + C_3 \]

Integrate again for y(t):

$$ y(t) = \int y'(t) dt + C_4 $$

$$ y(t) = C_1 \frac{-\cos(\sqrt{2} t)}{2} - C_2 \frac{\sin(\sqrt{2} t)}{2} + C_3 t + C_4 $$

But to be precise, integrating these trigonometric functions:

$$ y(t) = C_1 \frac{-\cos(\sqrt{2} t)}{2} - C_2 \frac{\sin(\sqrt{2} t)}{2} + C_3 t + C_4 $$

Note: constants C_1, C_2, C_3, C_4 are arbitrary constants.

Thus, the homogeneous solution is:

$$ y_h(t) = A \cos(\sqrt{2} t) + B \sin(\sqrt{2} t) + C t + D $$

where A, B, C, D are constants related to C_1, C_2, C_3, C_4.

Step 3: Find a Particular Solution

Next, we find a particular solution $ y_p(t) $ to the nonhomogeneous equation:

$$ y^{(4)} + 2 y'' = 3t + 6 $$

Since the nonhomogeneous part is a polynomial (degree 1), we try a polynomial of degree 3 for $ y_p(t) $:

$$ y_p(t) = a t^3 + b t^2 + c t + d $$

Calculate derivatives:

- $ y_p' = 3 a t^2 + 2 b t + c $
- $ y_p'' = 6 a t + 2 b $
- $ y_p^{(3)} = 6 a $
- $ y_p^{(4)} = 0 $

Substitute into the differential equation:

$$ y^{(4)} + 2 y'' = 0 + 2 (6 a t + 2 b) = 12 a t + 4 b $$

Set equal to RHS:

$$ 12 a t + 4 b = 3 t + 6 $$

Matching coefficients:

- For t:

$$ 12 a = 3 \implies a = \frac{3}{12} = \frac{1}{4} $$

- For constants:

$$ 4 b = 6 \implies b = \frac{6}{4} = \frac{3}{2} $$

Constants c and d do not appear in the derivatives and thus are not constrained by the equation. To determine them, we can consider the initial conditions later.

Therefore, the particular solution is:

$$ y_p(t) = \frac{1}{4} t^3 + \frac{3}{2} t^2 + c t + d $$

Step 4: Write the General Solution

The general solution combines the homogeneous and particular solutions:

$$ y(t) = y_h(t) + y_p(t) $$

$$ y(t) = A \cos(\sqrt{2} t) + B \sin(\sqrt{2} t) + C t + D + \frac{1}{4} t^3 + \frac{3}{2} t^2 + c t + d $$

Since C and c are both constants in the general solution, we can combine them:

$$ y(t) = A \cos(\sqrt{2} t) + B \sin(\sqrt{2} t) + (C + c) t + (D + d) + \frac{1}{4} t^3 + \frac{3}{2} t^2 $$

But to match initial conditions, we need to keep track of all constants separately.

Step 5: Apply Initial Conditions to Determine Constants

Recall the initial conditions:

- $y(0) = 0$
- $y'(0) = 0$
- $y''(0) = 1$
- $y^{(3)}(0) = 1$

Compute derivatives of $y(t)$:

- First derivative:

$$y'(t) = -A \sqrt{2} \sin(\sqrt{2} t) + B \sqrt{2} \cos(\sqrt{2} t) + (C + c) + \frac{3}{4} t^2 + 3 t$$

- Second derivative:

$$y''(t) = -A (\sqrt{2})$$

Frequently Asked Questions

What is the initial value problem given in the question?

The initial value problem is Y(4) = 2y'', with the differential equation Y = 3t + 6, and initial conditions y(0)=0, y'(0)=0, y''(0)=1, y'''(0)=1.

How do we interpret the differential equation Y = 3t + 6?

The equation suggests that Y is a function related to t, likely representing derivatives of y, but the notation appears inconsistent. Assuming Y represents y''(t), the differential equation is y'' = 3t + 6.

What are the steps to solve the initial value problem y'' = 3t + 6?

First, integrate y'' = 3t + 6 to find y', then integrate y' to find y, applying initial conditions to determine constants.

How do we apply initial conditions y(0)=0, y'(0)=0, y''(0)=1, y'''(0)=1 in the solution?

We use these initial conditions to solve for the constants of integration at each step of integration to ensure the solution matches the given values at t=0.

What is the solution for y''(t) given the initial condition y''(0)=1?

Integrating $y''=3t+6$, with $y''(0)=1$, yields $y'(t)= (3/2)t^2 + 6t + C1$, where C1 is found using $y'(0)=0$.

What is the complete solution for y(t) after integrating y'(t)?

Integrate $y'(t)$ to obtain $y(t)$, incorporating constants from integration and applying $y(0)=0$ to find the specific solution.

What is the significance of the condition Y(4)=2y'' in the problem?

This condition likely relates Y to y'' at $t=4$, providing an additional equation to determine any remaining constants or verify the solution.

What is the final expression for y(t) satisfying all initial conditions and the differential equation?

The final solution is $y(t) = (1/2)t^3 + 3t^2 + C$, with constants determined by initial conditions, resulting in $y(t)= (1/2)t^3 + 3t^2$ where applicable.

Additional Resources

Find The Solution Of Given Initial Value Problem Y(4) 2y'' Y=3t 6; Y(0)=y'(0)=0,y''(0)=y^(3)(0)=1

In the realm of differential equations, initial value problems (IVPs) serve as foundational tools to model real-world phenomena—from physics and engineering to biology and economics. Today, we delve into a particularly intriguing IVP:

Find the solution of the initial value problem where $Y^{(4)}(t) - 2 y''(t) + Y(t) = 3t + 6$, with initial conditions $Y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, and $y^{(3)}(0) = 1$.

This problem combines higher-order derivatives, non-homogeneous terms, and multiple initial conditions, making it a comprehensive case study in solving differential equations. Let's break down the process step by step, exploring the mathematical principles involved while maintaining clarity and accessibility.

Understanding the Problem: Breaking Down the Differential Equation

The initial step in tackling any differential equation is to understand its structure. The given problem appears to involve a fourth-order differential equation with initial conditions specified at $t=0$.

The equation is:

$$Y^{(4)}(t) - 2 y''(t) + Y(t) = 3t + 6$$

with initial conditions:

$$
\begin{cases}
Y(0) = 0, \\
y'(0) = 0, \\
y''(0) = 1, \\
y^{(3)}(0) = 1.
\end{cases}
$$

Note: The notation suggests that $Y(t)$ and $y(t)$ are related functions, possibly the same, but the problem explicitly mentions Y and y separately. For clarity, and based on typical differential equation notation, we'll assume $Y(t)$ is the primary function to solve for, and the derivatives are with respect to t.

Key observations:

- The equation involves derivatives up to the fourth order.
- The right-hand side is a linear function of t: $3t + 6$.
- The initial conditions involve the function Y and its derivatives at $t=0$.

Step 1: Recognizing the Form and Strategy

Given the structure, the solution involves two main parts:

1. Homogeneous solution ($Y_h(t)$): solving the associated homogeneous differential equation.

2. Particular solution ($Y_p(t)$): finding a specific solution to the non-homogeneous equation.

The general solution will be:

$$Y(t) = Y_h(t) + Y_p(t).$$

Step 2: Formulating the Homogeneous Equation

The homogeneous counterpart of the differential equation is:

$$ Y^{(4)}(t) - 2 y''(t) + Y(t) = 0. $$

Assuming $Y(t)$ is the same as $y(t)$, the homogeneous equation simplifies to:

$$ Y^{(4)} - 2 Y'' + Y = 0. $$

This is a linear, constant-coefficient differential equation.

Step 3: Solving the Homogeneous Equation

Step 3.1: Write the characteristic equation

Replace derivatives with powers of r:

$$ r^4 - 2 r^2 + 1 = 0. $$

Step 3.2: Solve the characteristic polynomial

The polynomial:

$$ r^4 - 2 r^2 + 1 = 0. $$

Let $s = r^2$, transforming it into a quadratic:

$$ s^2 - 2s + 1 = 0. $$

Solve for s:

$$ s^2 - 2s + 1 = 0 \Rightarrow (s - 1)^2 = 0. $$

Therefore,

$$ s = 1 \quad \text{(double root)}. $$

Back to r:

$$ r^2 = 1 \Rightarrow r = \pm 1. $$

Since s has multiplicity 2, the roots for r are:

- $r = 1$ (double root),
- $r = -1$ (double root).

Step 3.3: Write the homogeneous solution

For each root, the general solution is:

$$
Y_h(t) = C_1 e^{t} + C_2 t e^{t} + C_3 e^{-t} + C_4 t e^{-t}.
$$

Step 4: Finding the Particular Solution

The non-homogeneous part is $3t + 6$, a polynomial of degree 1. When the right side is polynomial, the particular solution is assumed to be a polynomial of the same degree:

$$
Y_p(t) = A t + B,
$$

where A, B are constants to be determined.

Step 4.1: Compute derivatives of $Y_p(t)$

$$
Y_p(t) = A t + B,
$$
$$
Y_p'(t) = A,
$$
$$
Y_p''(t) = 0,
$$
$$
Y_p^{(3)}(t) = 0,
$$
$$
Y_p^{(4)}(t) = 0.
$$

Step 4.2: Substitute into the original differential equation

Recall the original equation:

$$ Y^{(4)} - 2 Y'' + Y = 3t + 6. $$

Plugging in:

$$
0 - 2 \times 0 + (A t + B) = 3t + 6.
$$

Simplifies to:

\[

$$
A t + B = 3t + 6.
$$

Matching coefficients:

$$
A = 3,
$$
$$
B = 6.
$$

Thus, the particular solution:

$$
Y_p(t) = 3 t + 6.
$$

Step 5: Constructing the General Solution

The general solution combines homogeneous and particular parts:

$$
Y(t) = C_1 e^{t} + C_2 t e^{t} + C_3 e^{-t} + C_4 t e^{-t} + 3 t + 6.
$$

Step 6: Applying Initial Conditions

Recall the initial conditions:

$$
Y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad y^{(3)}(0) = 1.
$$

Assuming $Y(t)$ is the same as $y(t)$, we'll differentiate $Y(t)$ to find the derivatives at $t=0$.

Step 7: Computing Derivatives at $t=0$

Function:

$$
Y(t) = C_1 e^{t} + C_2 t e^{t} + C_3 e^{-t} + C_4 t e^{-t} + 3 t + 6.
$$

Derivative Calculations:

First derivative:

$$
Y'(t) = C_1 e^{t} + C_2 \left(e^{t} + t e^{t} \right) + C_3 (- e^{-t}) + C_4 \left(e^{-t} - t e^{-t} \right) + 3.
$$

Simplify:

$$
Y'(t) = C_1 e^{t} + C_2 e^{t} + C_2 t e^{t} - C_3 e^{-t} + C_4 e^{-t} - C_4 t e^{-t} + 3.
$$

At $t=0$:

$$
Y'(0) = C_1 (1) + C_2 (1) + 0 - C_3 (1) + C_4 (1) - 0 + 3,
$$
$$
Y'(0) = C_1 + C_2 - C_3 + C_4 + 3.
$$

Given $y'(0) = 0$, we have:

$$
C_1 + C_2 - C_3 + C_4 + 3 = 0. \quad (1)
$$

Second derivative:

Differentiate $Y'(t)$:

$$
Y''(t) = C_1 e^{t} + C_2 \left(e^{t} + e^{t} + t e^{t} \right) + C_3 e^{-t} + C_4 \left(- e^{-t} - e^{-t} + t e^{-t} \right),
$$

which simplifies to:

$$
Y''(t) = C_1 e^{t} + C_2 (2 e^{t} + t e^{t}) + C_3 e^{-t} + C_4 (-2 e^{-t} + t e^{-t}).
$$

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At \( t
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Find The Solution Of Given Initial Value Problem Y 4 2y Y 3t 6 Y 0 Y 0 0 Y 0 Y 3 0 1

Find The Solution Of Given Initial Value Problem Y 4 2y Y 3t 6 Y 0 Y 0 0 Y 0 Y 3 0 1

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