Find The Surface Area Of The Part Of The Paraboloid Y=x2+z2 That Lies Inside The Cylinder X2+z2=16.

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Understanding how to compute the surface area of a paraboloid segment within a cylindrical boundary is a fundamental problem in multivariable calculus. This task involves parametrizing the surface, setting appropriate limits, and applying the surface area formula. In this article, we will walk through the process step-by-step, providing detailed explanations and insights to help you master this topic.

Introduction to the Problem

The problem involves calculating the surface area of a specific part of a paraboloid defined by the equation:

$$ y = x^2 + z^2 $$

which is bounded above by a cylinder described by:

$$ x^2 + z^2 = 16 $$

This cylinder is a circular cylinder with radius 4, extending infinitely in the y-direction. The paraboloid opens upward, and we're interested in the portion of its surface that lies within the cylinder.

Key aspects:
- The paraboloid surface: $y = x^2 + z^2$
- The bounding cylinder: $x^2 + z^2 = 16$

Our goal: Find the surface area of the paraboloid inside the cylinder.

Understanding the Geometry

Before approaching the calculus, visualizing the problem helps:

- The paraboloid opens upward, with its vertex at the origin.
- The cylinder $x^2 + z^2 = 16$ slices through the paraboloid, creating a closed curve in the xy-z space.
- The intersection occurs where $y = x^2 + z^2$ and $x^2 + z^2 = 16$,

which implies at the boundary $y = 16$.

Thus, the portion of the paraboloid inside the cylinder is bounded from below at the vertex (0,0,0) and from above at the intersection with the cylinder, which occurs at $y = 16$.

Note: Since the paraboloid is symmetric and the boundary is a circle of radius 4, cylindrical coordinates are a natural choice.

Choosing Coordinates and Setting Up the Integral

Using Cylindrical Coordinates

Cylindrical coordinates are defined as:

$$
x = r \cos \theta,\quad z = r \sin \theta,\quad y = y
$$

with $r \geq 0$, $0 \leq \theta < 2\pi$.

- The boundary of the cylinder: $r = 4$
- The paraboloid: $y = x^2 + z^2 = r^2$

Thus, the portion of the paraboloid inside the cylinder corresponds to:

$$
0 \leq r \leq 4,\quad 0 \leq \theta < 2\pi,\quad y = r^2
$$

Surface Element of a Paraboloid

The surface element dS on a surface $y = f(r,\theta)$ can be derived from the parameterization:

$$
\mathbf{r}(r, \theta) = (r \cos \theta, r^2, r \sin \theta)
$$

The partial derivatives are:

$$
\mathbf{r}_r = (\cos \theta, 2r, \sin \theta)
$$
$$
\mathbf{r}_\theta = (-r \sin \theta, 0, r \cos \theta)
$$

The surface element is:

$$
dS = |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta
$$

Computing the Cross Product

$$
\mathbf{r}_r \times \mathbf{r}_\theta =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & 2r & \sin \theta \\
- r \sin \theta & 0 & r \cos \theta
\end{vmatrix}
$$

Calculating determinant:

$$
\mathbf{i} (2r \cdot r \cos \theta - \sin \theta \cdot 0) - \mathbf{j} (\cos \theta \cdot r \cos \theta - \sin \theta \cdot (- r \sin \theta)) + \mathbf{k} (\cos \theta \cdot 0 - 2r \cdot (- r \sin \theta))
$$

Simplify each component:

- $\mathbf{i}$: $2r^2 \cos \theta$
- $\mathbf{j}$: $- (\cos \theta \cdot r \cos \theta + r \sin^2 \theta) = - (r \cos^2 \theta + r \sin^2 \theta) = - r (\cos^2 \theta + \sin^2 \theta) = - r$
- $\mathbf{k}$: $2r^2 \sin \theta$

Thus,

$$
\mathbf{r}_r \times \mathbf{r}_\theta = (2 r^2 \cos \theta, - r, 2 r^2 \sin \theta)
$$

The magnitude:

$$
|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{ (2 r^2 \cos \theta)^2 + (-r)^2 + (2 r^2 \sin \theta)^2 }
$$

Simplify:

$$
= \sqrt{ 4 r^4 \cos^2 \theta + r^2 + 4 r^4 \sin^2 \theta }

\]

Note that:

\[
4 r^4 (\cos^2 \theta + \sin^2 \theta) + r^2 = 4 r^4 + r^2
\]

Therefore:

\[
|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{ 4 r^4 + r^2 } = \sqrt{ r^2 (4 r^2 + 1) } = r \sqrt{ 4 r^2 + 1 }
\]

Surface Area Element

The differential surface area element:

\[
dS = |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta = r \sqrt{ 4 r^2 + 1 } \, dr\, d\theta
\]

Limits of Integration

- \(r \) varies from 0 to 4 (the radius of the cylinder)
- \(\theta \) varies from 0 to \(2\pi \)

Since the paraboloid surface is above the xy-plane and within the cylinder, the limits are straightforward.

Formulating the Surface Area Integral

The total surface area \(S \) is:

\[
S = \int_{0}^{2\pi} \int_{0}^{4} r \sqrt{ 4 r^2 + 1 }\, dr\, d\theta
\]

Because the integrand does not depend on \(\theta \), the integral simplifies:

\[
S = 2 \pi \int_{0}^{4} r \sqrt{ 4 r^2 + 1 }\, dr
\]

Our main task reduces to evaluating:

$$
I = \int_{0}^{4} r \sqrt{ 4 r^2 + 1 }\, dr
$$

Calculating the Inner Integral

Substitution Method

Let:

$$
u = 4 r^2 + 1
$$

Then:

$$
du = 8 r\, dr \Rightarrow r\, dr = \frac{du}{8}
$$

When $r = 0$:

$$
u = 1
$$

When $r = 4$:

$$
u = 4 \times 16 + 1 = 64 + 1 = 65
$$

Express the integral:

$$
I = \int_{u=1}^{65} \sqrt{u} \cdot \frac{du}{8}
$$

Simplify:

$$
I = \frac{1}{8} \int_{1}^{65} u^{1/2}\, du
$$

Integrate:

$$
\int u^{1/2}\, du = \frac{2}{3} u^{3/2}
$$

Therefore:

\[
I = \frac{1}{8} \times \frac{2}{3} [u^{3/2}]_{1}^{65} = \frac{1}{12} \left[
65^{3/2} - 1^{3/2} \right]
\]

Note that:

\[
65^{3/2} = (65^{1/2})^3 = (\sqrt{65})^3
\]

Hence,

\[
I = \frac{1}{12} \left((\sqrt{65})^3 - 1 \right)
\]

Final Expression for Surface Area

Putting it all together:

\[
S = 2 \pi \times I = 2 \pi \times \frac{1}{12} \left((\sqrt{65})^3 -

Frequently Asked Questions

What is the surface area formula for the part of the paraboloid $y = x^2 + z^2$ inside the cylinder $x^2 + z^2 = 16$?

The surface area is calculated by setting up a double integral over the region inside the cylinder, using the surface element $dS = \sqrt{1 + (\partial y/\partial x)^2 + (\partial y/\partial z)^2}\, dx\, dz$, with $y = x^2 + z^2$, and integrating over $x^2 + z^2 \leq 16$.

How do you parameterize the surface $y = x^2 + z^2$ inside the cylinder $x^2 + z^2 = 16$?

A suitable parameterization uses polar coordinates: $x = r \cos \theta$, $z = r \sin \theta$, $y = r^2$, with r from 0 to 4 and θ from 0 to 2π, to cover the region inside the cylinder.

What is the value of the partial derivatives needed

for the surface area calculation?

The partial derivatives are $\partial y/\partial x = 2x$ and $\partial y/\partial z = 2z$, which are used to compute the surface element's magnitude.

How do you set up the integral for finding the surface area in this problem?

The surface area $S = \iint_D \sqrt{1 + (\partial y/\partial x)^2 + (\partial y/\partial z)^2}\, dx\, dz$, which becomes an integral over the disk $x^2 + z^2 \leq 16$, substituting the derivatives and converting to polar coordinates.

What is the final expression for the surface area after converting to polar coordinates?

The surface area is given by $S = \int_0^{2\pi} \int_0^{4} r \sqrt{1 + 4r^2}\, dr\, d\theta$, where the integrand accounts for the derivatives of y and the Jacobian r from the polar coordinate transformation.

How do you evaluate the integral for the surface area to obtain the numerical value?

First, perform the inner integral with respect to r: $\int_0^{4} r \sqrt{1 + 4r^2}\, dr$, then multiply by 2π. The inner integral can be evaluated using substitution $u = 1 + 4r^2$, leading to a straightforward calculation.

Additional Resources

Find The Surface Area Of The Part Of The Paraboloid $Y=x^2+z^2$ That Lies Inside The Cylinder $x^2+z^2=16$

Understanding the intricacies of three-dimensional surfaces is a fundamental aspect of advanced calculus, with applications spanning physics, engineering, and computer graphics. Today, we delve into a specific problem that combines the elegance of paraboloids and cylinders, two common surfaces in multivariable calculus. Our goal is to determine the surface area of a portion of a paraboloid—namely, the surface described by \(y = x^2 + z^2 \)—that lies within a given cylindrical boundary \(x^2 + z^2 = 16 \).

This problem not only offers a rich exploration of surface calculus but also provides insight into the methods used to handle complex surface regions via parametrization and integration. By the end of this article, you'll have a clear understanding of how to approach such problems step-by-step, from setting up the surface equations to performing the integral calculations necessary to find the surface area.

Understanding the Geometric Surfaces Involved

The Paraboloid $y = x^2 + z^2$

The paraboloid described by $y = x^2 + z^2$ opens upward in the y-direction, with its vertex at the origin. Its shape is symmetric around the y-axis, and it can be visualized as a smooth bowl extending infinitely in the positive y-direction.

This surface is a classic example of a quadratic surface, and its cross-sections parallel to the xz-plane are circles with radius $\sqrt{y}$. As y increases, the cross-sectional circle becomes larger, illustrating the paraboloid's expanding nature.

The Cylinder $x^2 + z^2 = 16$

The cylinder in question is a circular cylinder aligned along the y-axis, with radius 4 (since $\sqrt{16} = 4$). Its walls extend infinitely along the y-direction, but in our problem, we are only interested in the part of the paraboloid that lies inside this cylinder—that is, where the points (x, y, z) satisfy $x^2 + z^2 \leq 16$.

Visualizing the Intersection and Region of Integration

The intersection of the paraboloid and the cylinder occurs where the paraboloid's points also satisfy the cylinder’s boundary $x^2 + z^2 \leq 16$. Substituting $y = x^2 + z^2$ into the boundary condition, we observe that:

$$
x^2 + z^2 \leq 16
$$

implies:

$$
y \leq 16
$$

meaning that the paraboloid's surface portion within the cylinder extends from the vertex at the origin up to the 'cap' defined by $y = 16$.

Thus, the region of interest is bounded between:

- The base at $y = 0$ (the vertex), and
- The 'top' at $y = 16$, where the paraboloid intersects the cylinder boundary.

The goal is to find the surface area of this portion of the paraboloid.

Setting Up the Surface Area Calculation

The Surface Area Element

In multivariable calculus, the surface area S of a surface parametrized by functions $x(u, v)$, $y(u, v)$, $z(u, v)$ can be computed using:

$$
S = \iint_D \left| \mathbf{r}_u \times \mathbf{r}_v \right| \, du \, dv
$$

Alternatively, for a surface expressed explicitly as $y = f(x, z)$, the differential surface element dS can be written as:

$$
dS = \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2 + \left(\frac{\partial y}{\partial z} \right)^2} \, dx \, dz
$$

This approach simplifies the problem because the surface $y = x^2 + z^2$ is given explicitly.

Calculating the Partial Derivatives

Given:

$$
y = x^2 + z^2
$$

the partial derivatives are:

$$
\frac{\partial y}{\partial x} = 2x
$$
$$
\frac{\partial y}{\partial z} = 2z
$$

Plugging into the surface element formula:

$$
dS = \sqrt{1 + (2x)^2 + (2z)^2} \, dx \, dz = \sqrt{1 + 4x^2 + 4z^2} \, dx \, dz
$$

Expressing the Region of Integration

The projection of the surface onto the xz-plane is within the disk:

$$
x^2 + z^2 \leq 16
$$

To simplify the integration, it is natural to switch to polar coordinates:

$$
x = r \cos \theta,\quad z = r \sin \theta
$$

with:

$$
r \in [0, 4], \quad \theta \in [0, 2\pi]
$$

In these coordinates:

$$
x^2 + z^2 = r^2
$$
$$
dx \, dz = r \, dr \, d\theta
$$

and the integrand becomes:

$$
\sqrt{1 + 4r^2}
$$

since:

$$
4x^2 + 4z^2 = 4r^2
$$

Setting Up the Integral for Surface Area

Putting it all together, the surface area S:

$$
S = \int_{0}^{2\pi} \int_{0}^{4} \sqrt{1 + 4r^2} \, r \, dr \, d\theta
$$

\]

Because the integrand is radially symmetric, the angular integral simplifies to multiplication by \(2\pi \):

\[
S = 2\pi \int_{0}^{4} r \sqrt{1 + 4r^2} \, dr
\]

Solving the Radial Integral

The key task now is to evaluate:

\[
I = \int_{0}^{4} r \sqrt{1 + 4r^2} \, dr
\]

This integral is suitable for substitution. Let’s define:

\[
u = 1 + 4r^2
\]

then:

\[
du = 8r \, dr
\]
\[
r \, dr = \frac{du}{8}
\]

When \(r = 0 \):

\[
u = 1 + 0 = 1
\]

When \(r = 4 \):

\[
u = 1 + 4 \times 16 = 1 + 64 = 65
\]

Rewriting the integral:

\[
I = \int_{u=1}^{65} \sqrt{u} \times \frac{du}{8}
\]

which simplifies to:

$$
I = \frac{1}{8} \int_{1}^{65} u^{1/2} \, du
$$

The integral:

$$
\int u^{1/2} \, du = \frac{2}{3} u^{3/2}
$$

Thus,

$$
I = \frac{1}{8} \times \frac{2}{3} \left[u^{3/2} \right]_1^{65} = \frac{1}{12} \left(65^{3/2} - 1^{3/2} \right)
$$

Recall that:

$$
65^{3/2} = (65^{1/2})^3 = (\sqrt{65})^3
$$

Hence,

$$
I = \frac{1}{12} \left((\sqrt{65})^3 - 1 \right)
$$

Final Computation of the Surface Area

The total surface area:

$$
S = 2\pi \times I = 2\pi \times \frac{1}{12} \left((\sqrt{65})^3 - 1 \right) = \frac{\pi}{6} \left((\sqrt{65})^3 - 1 \right)
$$

Expressed explicitly:

$$
\boxed{
\text{Surface Area} = \frac{\pi}{6} \left((\sqrt{65})^3 - 1 \right)
}
$$

This formula provides the exact surface area of the portion of the paraboloid

$y = x^2 + z^2$ lying inside the cylinder $x^2 + z^2 = 16$, from the vertex at $y=0$ up to the intersection at $y=16$.

Intuitive Insights and Applications

The calculation of this surface area underscores several key techniques in multivariable calculus:

- Switching to polar coordinates simplifies integration over circular regions.
- Substitution methods are essential for integrals involving radicals and quadratic expressions.
- Understanding the geometry of the surfaces helps determine appropriate

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