Find The Volume Of The Given Solid. Bounded By The Coordinate Planes And The Plane 8x + 9y + Z = 72

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Understanding the volume of a solid bounded by coordinate planes and a plane is a fundamental problem in multivariable calculus. It involves integrating over a three-dimensional region defined by inequalities, which can be approached systematically using methods such as triple integrals and coordinate transformations. In this article, we will explore the step-by-step process to find the volume of the solid bounded by the coordinate planes $x=0$, $y=0$, $z=0$, and the plane $8x + 9y + z = 72$.

Understanding the Geometric Region

Before we delve into calculations, it's essential to visualize the region whose volume we are calculating.

The Boundary Planes

The solid is bounded by:

- The xy-plane: $z=0$
- The yz-plane: $x=0$
- The xz-plane: $y=0$
- The plane $8x + 9y + z = 72$

The first three are coordinate planes, which form the lower bounds in each respective axis.

The Plane $8x + 9y + z = 72$

This plane intersects the coordinate axes at points where the other variables are zero:

- When $y=0$ and $z=0$, $8x=72 \Rightarrow x=9$
- When $x=0$ and $z=0$, $9y=72 \Rightarrow y=8$
- When $x=0$ and $y=0$, $z=72$

The intercepts are therefore:

- x-intercept at $(9, 0, 0)$
- y-intercept at $(0,8, 0)$
- z-intercept at $(0, 0, 72)$

This indicates the plane cuts through the first octant creating a bounded region with the coordinate planes.

Setting Up the Integral to Find the Volume

The volume V of the solid can be expressed as a triple integral over the region R:

$$
V = \iiint_{R} dV
$$

where R is the region bounded by the coordinate planes and the plane $8x + 9y + z=72$.

Expressing the Region R

Since the boundary is given by the plane, it’s convenient to express z in terms of x and y:

$$
z = 72 - 8x - 9y
$$

Given that $z \geq 0$ within the solid, the limits for z are:

$$
0 \leq z \leq 72 - 8x - 9y
$$

For x and y, the limits are determined by the conditions:

$$
8x + 9y \leq 72
$$

with $x, y \geq 0$.

Determining the Limits for x and y

The region in the xy-plane is bounded by:

$$
8x + 9y \leq 72
$$

and both $x, y \geq 0$.

- For a fixed x, the maximum y is:

$$
9y \leq 72 - 8x \Rightarrow y \leq \frac{72 - 8x}{9}
$$

- The maximum x occurs when $y=0$:

$$
8x \leq 72 \Rightarrow x \leq 9
$$

Therefore, the limits are:

$$
0 \leq x \leq 9
$$
$$
0 \leq y \leq \frac{72 - 8x}{9}
$$
$$
0 \leq z \leq 72 - 8x - 9y
$$

Formulating the Triple Integral

Putting everything together, the volume integral becomes:

$$
V = \int_{x=0}^{9} \int_{y=0}^{\frac{72 - 8x}{9}} \int_{z=0}^{72 - 8x - 9y} dz\, dy\, dx
$$

Since the integrand is simply 1 (we are calculating volume), the integral simplifies to:

$$
V = \int_{0}^{9} \int_{0}^{\frac{72 - 8x}{9}} \left[z \right]_{0}^{72 - 8x - 9y} dy\, dx
$$

$$
V = \int_{0}^{9} \int_{0}^{\frac{72 - 8x}{9}} (72 - 8x - 9y) dy\, dx

\]

Next, perform the integration with respect to \(y\).

Integrating with Respect to \(y\)

\[
V = \int_{0}^{9} \left[\int_{0}^{\frac{72 - 8x}{9}} (72 - 8x - 9y) dy \right] dx
\]

Compute the inner integral:

\[
\int_{0}^{Y_{max}} (A - 9y) dy
\]
where \(A = 72 - 8x\) and \(Y_{max} = \frac{72 - 8x}{9}\).

This leads to:

\[
\int_{0}^{Y_{max}} (A - 9y) dy = \left[A y - \frac{9 y^2}{2} \right]_0^{Y_{max}}
\]

Substitute \(Y_{max}\):

\[
A \cdot Y_{max} - \frac{9}{2} (Y_{max})^2
\]

Recall that \(A = 72 - 8x\), so:

\[
V = \int_{0}^{9} \left[(72 - 8x) \frac{72 - 8x}{9} - \frac{9}{2} \left(\frac{72 - 8x}{9}\right)^2 \right] dx
\]

Simplify each term:

- First term:

\[
(72 - 8x) \frac{72 - 8x}{9} = \frac{(72 - 8x)^2}{9}
\]

- Second term:

\[

$$
\frac{9}{2} \left(\frac{72 - 8x}{9}\right)^2 = \frac{9}{2} \cdot \frac{(72 - 8x)^2}{81} = \frac{(72 - 8x)^2}{18}
$$

Putting these together:

$$
V = \int_{0}^{9} \left[\frac{(72 - 8x)^2}{9} - \frac{(72 - 8x)^2}{18} \right] dx
$$

Factor out $(72 - 8x)^2$:

$$
V = \int_{0}^{9} (72 - 8x)^2 \left(\frac{1}{9} - \frac{1}{18} \right) dx
$$

Calculate the difference inside parentheses:

$$
\frac{1}{9} - \frac{1}{18} = \frac{2}{18} - \frac{1}{18} = \frac{1}{18}
$$

Therefore:

$$
V = \frac{1}{18} \int_{0}^{9} (72 - 8x)^2 dx
$$

Now, expand $(72 - 8x)^2$:

$$
(72 - 8x)^2 = 72^2 - 2 \times 72 \times 8x + (8x)^2 = 5184 - 1152x + 64x^2
$$

The volume becomes:

$$
V = \frac{1}{18} \int_{0}^{9} (5184 - 1152x + 64x^2) dx
$$

Integrate term by term:

$$
V = \frac{1}{18} \left[5184x - 576x^2 + \frac{64}{3} x^3 \right]_0^{9}
$$

Evaluate at $x=9$:

- $5184 \times 9 = 46656$
- $576 \times 81 = 46656$

- \(\frac{64}{3} \times 729 = \frac{64 \times 729}{3} = \frac{46656}{3} = 15552\)

Putting it all together:

\[
V = \frac{1}{18} \left(46656 - 46656 + 15552 \right) = \frac{1}{18} \times 15552
\]

Simplify:

\[
V = \frac{15552}{18}

Frequently Asked Questions

What is the solid bounded by the coordinate planes and the plane 8x + 9y + z = 72?

It is a tetrahedral region in the first octant limited by the coordinate planes (x=0, y=0, z=0) and the plane 8x + 9y + z = 72.

How do you find the intercepts of the plane 8x + 9y + z = 72 with the coordinate axes?

Set two variables to zero to find each intercept: x-intercept at (9,0,0), y-intercept at (0,8,0), and z-intercept at (0,0,72).

What is the volume of the solid bounded by the coordinate planes and the plane 8x + 9y + z = 72?

The volume is given by V = (1/6) (x_intercept) (y_intercept) (z_intercept) = (1/6) 9 8 72 = 864 cubic units.

How can the volume of this tetrahedron be calculated using a triple integral?

Set up the integral as $V = \iiint dx\ dy\ dz$ over the region bounded by the coordinate planes and the plane, integrating in an order such as dz dy dx, with limits determined by the plane equations.

What are the limits of integration for x, y, and z

in the triple integral for this solid?

x from 0 to 9, y from 0 to (8 - (8/9)x), and z from 0 to (72 - 8x - 9y).

How do you evaluate the volume integral for this solid?

Perform the integration step by step: integrate with respect to z, then y, then x, substituting the limits at each step to find the total volume.

Can the volume be determined using geometric formulas instead of integrals?

Yes, since it is a tetrahedron, the volume formula V = (1/6) base area height can be used, which matches the coordinate intercepts calculation.

What is the significance of the intercepts in calculating the volume of this solid?

The intercepts define the vertices of the tetrahedron, allowing direct calculation of volume using the formula for a pyramid or tetrahedron.

Are there any similar problems involving finding the volume of solids bounded by planes and coordinate axes?

Yes, many problems involve calculating the volume of tetrahedra or other polyhedra bounded by planes and axes, often using intercepts or triple integrals for more complex shapes.

Additional Resources

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Introduction

Understanding the volume of three-dimensional solids is a fundamental aspect of analytical geometry and calculus, providing insights into how spatial regions are quantified. When a solid is bounded by coordinate planes and a plane equation, its volume can be determined through integration techniques, offering both theoretical and practical applications across sciences and engineering. In this article, we will explore the process of calculating the volume of a specific solid bounded by the coordinate planes (the x-y, y-z, and x-z planes) and the plane given by the equation $8x + 9y + z = 72$. This problem exemplifies the application of triple integrals in a three-dimensional context, illustrating the step-by-step approach to solving such geometric problems with clarity and depth.

Understanding the Geometric Configuration

The Boundaries of the Solid

The problem states that the solid is bounded by:

- The coordinate planes: $x = 0$, $y = 0$, and $z = 0$.
- The plane: $8x + 9y + z = 72$.

This configuration creates a three-dimensional region in the first octant, where all coordinates are non-negative. The plane intersects the coordinate axes at specific points, which define the limits of the region.

Geometric Significance

The bounded region is a tetrahedral volume with vertices at the intercepts of the plane with the axes:

- x-intercept: Set $y=0$, $z=0$ in the plane:
$8x + 0 + 0 = 72 \rightarrow x = 9$.

- y-intercept: Set $x=0$, $z=0$:
$0 + 9y + 0 = 72 \rightarrow y = 8$.

- z-intercept: Set $x=0$, $y=0$:
$0 + 0 + z = 72 \rightarrow z = 72$.

Thus, the tetrahedron is bounded by the coordinate planes and the plane connecting these intercepts.

Mathematical Approach to the Volume Calculation

Step 1: Expressing the Plane Equation

To facilitate integration, express z in terms of x and y:

$$ z = 72 - 8x - 9y $$

This expression indicates that for any point in the volume, z varies from 0 (the xy-plane) up to the plane $8x + 9y + z = 72$.

Step 2: Determining the Limits of Integration

The feasible region in the xy-plane is dictated by the positivity constraints:

- Since $z \geq 0$, it follows that:

$$ 72 - 8x - 9y \geq 0 $$
$$ 8x + 9y \leq 72 $$

- Both x and y are ≥ 0.

This inequality defines a triangular region in the xy-plane with vertices at:

- (0, 0): when x=0, y=0.
- (9, 0): when y=0, x=9.
- (0, 8): when x=0, y=8.

These points are consistent with the intercepts of the plane.

Step 3: Setting Up the Triple Integral

The volume V can be calculated via the triple integral:

$$ V = \iiint_{D} dV $$

Where D is the region bounded by the coordinate planes and the plane.

Expressed in iterated integral form, considering the order dz dy dx:

$$ V = \int_{x=0}^{9} \int_{y=0}^{(72 - 8x)/9} \int_{z=0}^{72 - 8x - 9y} dz \, dy \, dx $$

Step-by-Step Calculation

Step 4: Integrating with Respect to z

The inner integral over z:

$$ \int_{z=0}^{72 - 8x - 9y} dz = 72 - 8x - 9y $$

This simplifies the triple integral to a double integral:

$$ V = \int_{x=0}^{9} \int_{y=0}^{(72 - 8x)/9} (72 - 8x - 9y) dy \, dx $$

Step 5: Integrating with Respect to y

Next, perform the integration over y:

$$ \int_{0}^{(72 - 8x)/9} (72 - 8x - 9y) dy $$

Let’s denote:

$$ A = 72 - 8x $$

Then the integral becomes:

$$ \int_{0}^{A/9} (A - 9y) dy $$

Calculating:

$$
\begin{aligned}
\int_{0}^{A/9} (A - 9y) dy &= \left[A y - \frac{9 y^2}{2} \right]_0^{A/9} \\
&= A \times \frac{A}{9} - \frac{9}{2} \times \left(\frac{A}{9} \right)^2 \\
&= \frac{A^2}{9} - \frac{9}{2} \times \frac{A^2}{81} \\
&= \frac{A^2}{9} - \frac{A^2}{18} \\
&= \left(\frac{2A^2}{18} - \frac{A^2}{18} \right) \\
&= \frac{A^2}{18}
\end{aligned}
$$

Recall that $A = 72 - 8x$, so:

$$ \int_{0}^{(72 - 8x)/9} (72 - 8x - 9y) dy = \frac{(72 - 8x)^2}{18} $$

Step 6: Integrating with Respect to x

The remaining integral:

$$ V = \int_{0}^{9} \frac{(72 - 8x)^2}{18} dx $$

Simplify the constant:

$$ V = \frac{1}{18} \int_{0}^{9} (72 - 8x)^2 dx $$

Expanding $(72 - 8x)^2$:

$$
(72 - 8x)^2 = 72^2 - 2 \times 72 \times 8x + (8x)^2 = 5184 - 1152 x + 64 x^2
$$

Thus:

$$
V = \frac{1}{18} \int_{0}^{9} (5184 - 1152 x + 64 x^2) dx
$$

Calculating each term separately:

$$
\begin{aligned}

V &= \frac{1}{18} \left[5184 x - 1152 \frac{x^2}{2} + 64 \frac{x^3}{3} \right]_0^{9} \\
&= \frac{1}{18} \left[5184 \times 9 - 1152 \times \frac{81}{2} + 64 \times \frac{729}{3} \right]
\end{aligned}
\]

Compute each term:

- \(5184 \times 9 = 46656 \)
- \(1152 \times \frac{81}{2} = 1152 \times 40.5 = 46716 \)
- \(64 \times \frac{729}{3} = 64 \times 243 = 15552 \)

Substitute back:

\[
V = \frac{1}{18} \left(46656 - 46716 + 15552 \right) = \frac{1}{18} \left(-60 + 15552 \right) = \frac{1}{18} \times 15552 - \frac{1}{18} \times 60
\]

Simplify:

\[
V = \frac{15552}{18} - \frac{60}{18} = 864 - \frac{10}{3}
\]

Expressed as an improper fraction:

\[
V = 864 - \frac{10}{3} = \frac{2592}{3} - \frac{10}{3} = \frac{2582}{3}
\]

Final Result and Interpretation

The calculated volume of the solid bounded by the coordinate planes and the plane 8x + 9y + z = 72 in the first octant is:

\[
\boxed{
V = \frac{2582}{3} \text{ cubic units}
}
\]

This result underscores the power of integral calculus in solving complex geometric problems, translating spatial boundaries into algebraic limits, and systematically evaluating the volume enclosed by these surfaces.

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Significance and Applications

Understanding how to compute such volumes is crucial in numerous real-world
applications, including:

- Engineering: Designing components with specific spatial constraints.
- Physics: Calculating mass or charge distributions within bounded regions.
- Environmental Science: Estimating volumes of landforms or pollutants
confined within certain boundaries.
- Economics and Optimization: Determining feasible regions for resource
allocation.

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