Find The Volume Of The Solid Bounded By The Xy- Plane And The Surfaces X + Y=144 And Z=x+y. ______

Find The Volume Of The Solid Bounded By The Xy- Plane And The Surfaces X + Y=144 And Z=x+y.

Understanding how to find the volume of a solid bounded by specific surfaces and planes is a fundamental problem in multivariable calculus. This type of problem involves setting up and evaluating a double integral over a region in the xy-plane, where the limits are defined by the given surfaces. In this article, we will explore the step-by-step process to determine the volume of the solid bounded by the xy-plane, the plane x + y = 144, and the surface z = x + y. We will cover the geometric interpretation, the setup of the integral, methods for evaluating it, and practical tips to ensure accurate calculations. This comprehensive guide aims to clarify the concepts and provide a thorough understanding suitable for students, educators, and anyone interested in calculus applications.

Understanding the Geometric Configuration

Before delving into the calculations, it’s essential to visualize the problem and understand the geometric shapes involved.

Surfaces and Planes Involved

- XY-plane (z=0): The base of the solid, lying flat in the xy-plane.
- Plane x + y = 144: A straight plane inclined in 3D space, intercepting axes at (144, 0, 0) and (0, 144, 0).
- Surface z = x + y: A plane that rises above the xy-plane, with z increasing linearly as x and y increase.

Region of Interest in the XY-Plane

The projection of the solid onto the xy-plane is a region bounded by:
- The coordinate axes (x=0, y=0).
- The line x + y = 144, which forms a straight boundary.

The region in the xy-plane is therefore a right triangle with vertices at:
- (0, 0)
- (144, 0)
- (0, 144)

This triangular region will be the domain of integration.

Visualizing the Solid

- The bottom surface is the xy-plane (z=0).
- The top surface is the plane z = x + y.
- The vertical boundaries are the coordinate axes and the line x + y = 144.

The solid resembles a three-dimensional wedge rising from the xy-plane up to the plane z = x + y, bounded horizontally by the triangle in the xy-plane.

Setting Up the Volume Integral

The key to solving the problem is to express the volume as a double integral over the region R in the xy-plane:

$$
V = \iint_R \text{(height)} \, dA
$$

where the height at each point (x, y) is given by the surface z = x + y, since the solid extends from the xy-plane (z=0) up to z = x + y.

Formulating the Integral

- The height of the solid at a point (x, y) is $z = x + y$.
- Therefore, the volume is:

$$
V = \iint_R (x + y) \, dA
$$

- The region R is the triangle bounded by:
- x = 0
- y = 0
- x + y = 144

Choosing the Order of Integration

Two common options are:
- Integrate with respect to y first, then x.
- Integrate with respect to x first, then y.

Given the boundary x + y = 144, it is convenient to integrate with respect to y first:

- For a fixed x, y varies from 0 to 144 - x.
- x varies from 0 to 144.

Thus, the limits are:

$$
0 \leq x \leq 144
$$

$$
0 \leq y \leq 144 - x
$$

The double integral becomes:

$$
V = \int_{x=0}^{144} \int_{y=0}^{144 - x} (x + y) \, dy \, dx
$$

Calculating the Double Integral

Now, we proceed to evaluate the integral step-by-step.

Inner Integral

$$
I_x = \int_{0}^{144 - x} (x + y) \, dy
$$

Since x is treated as a constant during the inner integration, we integrate:

$$
I_x = \int_{0}^{144 - x} x \, dy + \int_{0}^{144 - x} y \, dy
$$

Calculating each term separately:

- $\int_{0}^{144 - x} x \, dy = x \cdot (144 - x)$

- $\int_{0}^{144 - x} y \, dy = \frac{1}{2} (144 - x)^2$

Therefore,

$$
I_x = x (144 - x) + \frac{1}{2} (144 - x)^2
$$

Outer Integral

The volume is then:

$$
V = \int_{0}^{144} \left[x (144 - x) + \frac{1}{2} (144 - x)^2 \right] dx
$$

Simplify the integrand:

$$
V = \int_{0}^{144} \left[144x - x^2 + \frac{1}{2} (144 - x)^2 \right] dx
$$

Evaluating the Integral

Break the integral into manageable parts:

$$
V = \int_{0}^{144} 144x \, dx - \int_{0}^{144} x^2 \, dx + \frac{1}{2} \int_{0}^{144} (144 - x)^2 \, dx
$$

Calculate each term separately.

First Term: $\int_{0}^{144} 144x \, dx$

$$
= 144 \cdot \frac{x^2}{2} \bigg|_{0}^{144} = 144 \cdot \frac{(144)^2}{2}
$$

Calculate:

$$
(144)^2 = 20736
$$

So,

$$
144 \cdot \frac{20736}{2} = 144 \times 10368 = 1,492,992
$$

Second Term: $\int_{0}^{144} x^2 \, dx$

$$
= \frac{x^3}{3} \bigg|_{0}^{144} = \frac{(144)^3}{3}
$$

Calculate 144^3:

$$
144^3 = 144 \times 144 \times 144 = (144 \times 144) \times 144 = 20736 \times 144
$$

$$
20736 \times 144 = (20000 + 736) \times 144 = 20000 \times 144 + 736 \times 144 = 2,880,000 + 106,304 = 2,986,304
$$

Now divide by 3:

$$
\frac{2,986,304}{3} = 995,434.67
$$

(here, exact fraction is preferable, but decimal approximation is fine for understanding).

Third Term: $\frac{1}{2} \int_{0}^{144} (144 - x)^2 dx$

Make the substitution:

$$
u = 144 - x \Rightarrow du = -dx
$$

When $x=0$, $u=144$; when $x=144$, $u=0$.

Reversing limits:

$$
\int_{0}^{144} (144 - x)^2 dx = \int_{u=144}^{0} u^2 (-du) = \int_{0}^{144} u^2 du
$$

Calculate:

$$
\int_{0}^{144} u^2 du = \frac{u^3}{3} \bigg|_{0}^{144} = \frac{(144)^3}{3} = 995,434.67
$$

Now multiply by $\frac{1}{2}$:

$$
\frac{1}{2} \times 995,434.67 \approx 497,717.33
$$

Summing It All Up

Now, combine the three parts:

$$
V = 1,492,992 - 995,434.67 + 497,717.33
$$

Calculate:

$$
V \approx (1,492,992 - 995,434.67) + 497,717.33 = 497,557.33 + 497,717.33 = 995,274.66
$$

The approximate volume of the solid is 995,275 cubic units.

Alternative Methods and Verification

While the above approach uses direct integration, alternative methods can include:

- Changing the order of integration: Swapping the order to integrate with respect to x first.
- Using symmetry: Recognizing the symmetry of the region to simplify calculations.
- Using software

Frequently Asked Questions

How do I set up the integral to find the volume of the solid bounded by the XY-plane, the surface X + Y = 144, and Z = X + Y?

You can set up a double integral over the region in the XY-plane bounded by $X \geq 0$, $Y \geq 0$, and $X + Y \leq 144$. Since $Z = X + Y$, the volume V is given by $V = \iint_R (X + Y)\, dA$, where R is the triangular region with vertices at (0,0), (144,0), and (0,144).

What are the limits of integration for calculating the volume in this problem?

Using the region bounded by X from 0 to 144 - Y, and Y from 0 to 144, the limits are Y from 0 to 144, and for each fixed Y, X from 0 to 144 - Y.

How can I evaluate the double integral for the volume step-by-step?

First, set up the integral: $V = \int_0^{144} \int_0^{144 - Y} (X + Y)\, dX\, dY$. Integrate with respect to X first: $\int_0^{144 - Y} (X + Y)\, dX = \left[(X^2/2) + YX \right]_0^{144 - Y}$. Then, integrate the resulting expression with respect to Y from 0 to 144.

What is the final calculated volume of the solid bounded by the given surfaces?

After performing the integrations, the volume V equals 1,728,000 cubic units.

Can this problem be solved using a different coordinate system, such as polar coordinates?

No, since the region is a right triangle bounded by straight lines in the XY-plane, Cartesian coordinates are more straightforward. Polar coordinates are more suitable for circular or radial regions.

Additional Resources

Find The Volume Of The Solid Bounded By The Xy- Plane And The Surfaces X + Y=144 And Z=x+y. ______

Understanding the problem of finding the volume of a solid bounded by specific surfaces is a fundamental aspect of multivariable calculus. This problem involves a three-dimensional shape defined by two surfaces and a plane, and determining its volume requires an integration approach that combines geometry and calculus. By exploring this problem, we can deepen our understanding of how to visualize and compute volumes bounded by surfaces, which has applications in fields ranging from physics and engineering to computer graphics and environmental modeling.

In this article, we will methodically analyze the problem, define the boundaries, set up the appropriate integrals, and then compute the volume step by step. Our goal is to present a clear, detailed, and reader-friendly explanation that balances technical rigor with intuitive understanding.

Understanding the Surfaces and Boundaries

Before diving into calculations, it’s essential to visualize and understand the surfaces that define the solid.

The XY-Plane

- The xy-plane is the horizontal plane where $z=0$.
- It acts as the lower boundary of the solid.

The Plane $X + Y = 144$

- This is a plane slanting across the xy-plane.
- It intercepts the axes at:
- $X=144, Y=0$
- $Y=144, X=0$
- The equation can be rewritten as:

$$
Y = 144 - X
$$

- This boundary forms an inclined "roof" over the xy-plane, creating a triangular region in the xy-plane where $X \geq 0$, $Y \geq 0$, and $X + Y \leq 144$.

The Surface $Z = X + Y$

- This surface is a plane inclined towards increasing Z, rising linearly with X and Y.
- It intersects the xy-plane at $Z=0$, which occurs when $X + Y=0$, i.e., at the origin.
- The surface extends upward over the region where $X + Y \leq 144$.

Visualizing the Solid

The solid bounded by these surfaces is essentially a three-dimensional shape:
- The bottom is the xy-plane ($z=0$).
- The top is the surface $z = x + y$.
- The lateral boundary is the plane $x + y = 144$.

Imagine a "triangular" region in the xy-plane bounded by:
- The axes $X=0$ and $Y=0$,
- The line $X + Y = 144$.

Over this region, the surface $z = x + y$ forms the "roof" of the solid, rising from the xy-plane up to the plane defined by $x + y = 144$.

Key insight: The volume is the space beneath the surface $z = x + y$, above the xy-region bounded by $X=0$, $Y=0$, and $X + Y = 144$.

Setting Up the Integral for Volume Calculation

To compute the volume, we employ a double integral over the xy-region, integrating the height function $z = x + y$:

$$
V = \iint_{D} (z_{\text{top}} - z_{\text{bottom}}) \, dA
$$

Since the bottom is the xy-plane ($z=0$), the volume simplifies to:

$$
V = \iint_{D} (x + y) \, dA
$$

where D is the projection of the solid onto the xy-plane, i.e., the triangular region:

$$
D = \{(x, y) \mid x \geq 0, \ y \geq 0, \ x + y \leq 144\}
$$

Choosing Limits of Integration:

- Option 1: Integrate with respect to y first, then x.

For each fixed x in $[0, 144]$, the variable y varies from 0 up to $144 - x$:

$$
y: 0 \leq y \leq 144 - x
$$

and

$$
x: 0 \leq x \leq 144
$$

- Option 2: Integrate with respect to x first, then y.

For each fixed y, x varies from 0 up to $144 - y$:

$$
x: 0 \leq x \leq 144 - y
$$

and

$$
y: 0 \leq y \leq 144
$$

Both approaches are valid; we'll proceed with the first for clarity.

Integral expression:

$$
V = \int_{x=0}^{144} \int_{y=0}^{144 - x} (x + y) \, dy \, dx
$$

Note: The integrand $x + y$ is linear, making the integration straightforward.

Calculating the Integral Step by Step

Let's now evaluate the double integral, step by step.

Inner Integral: Integrate with respect to y

$$
I_1(x) = \int_{0}^{144 - x} (x + y) \, dy
$$

Since x is treated as a constant during this integration:

$$
I_1(x) = \int_{0}^{144 - x} x \, dy + \int_{0}^{144 - x} y \, dy
$$

Calculating each:

$$
\int_{0}^{144 - x} x \, dy = x \cdot (144 - x)
$$

$$
\int_{0}^{144 - x} y \, dy = \left[\frac{y^2}{2} \right]_0^{144 - x} = \frac{(144 - x)^2}{2}
$$

Therefore,

$$
I_1(x) = x (144 - x) + \frac{(144 - x)^2}{2}
$$

Simplify $I_1(x)$:

$$
I_1(x) = 144x - x^2 + \frac{(144 - x)^2}{2}
$$

Express $(144 - x)^2$:

$$
(144 - x)^2 = 144^2 - 2 \times 144 \times x + x^2 = 20736 - 288x + x^2
$$

Plug back into $I_1(x)$:

$$
I_1(x) = 144x - x^2 + \frac{20736 - 288x + x^2}{2}
$$

Distribute the $\frac{1}{2}$:

$$
I_1(x) = 144x - x^2 + \frac{20736}{2} - \frac{288x}{2} + \frac{x^2}{2}
$$

Simplify:

$$
I_1(x) = 144x - x^2 + 10368 - 144x + \frac{x^2}{2}
$$

Notice $144x - 144x = 0$, so they cancel out:

$$
I_1(x) = - x^2 + 10368 + \frac{x^2}{2}
$$

Combine like terms:

$$
I_1(x) = 10368 + \left(- x^2 + \frac{x^2}{2} \right) = 10368 - \frac{x^2}{2}
$$

Inner integral result:

$$
I_1(x) = 10368 - \frac{x^2}{2}
$$

Outer Integral: Integrate with respect to x

Now, the volume:

$$
V = \int_{0}^{144} I_1(x) \, dx = \int_{0}^{144} \left(10368 - \frac{x^2}{2} \right) dx
$$

Break into two integrals:

$$
V = \int_0^{144} 10368 \, dx - \frac{1}{2} \int_0^{144} x^2 \, dx
$$

Calculate:

$$
\int_0^{144} 10368 \, dx = 10368 \times 144
$$

$$
\int_0^{144} x^2 \, dx = \left[\frac{x^3}{3} \right]_0^{144} = \frac{(144)^3}{3}
$$

Putting it together:

$$
V = 10368 \times 144 - \frac{1}{2} \times \frac{(144)^3}{3}
$$

Simplify:

$$
V = 10368 \times 144 - \frac{(144)^3}{6}
$$

Calculating

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