Find The Volume Of The Tetrahedron Bounded By The Coordinate Planes And The Plane X+2y+56z=45 _____

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Understanding how to determine the volume of a tetrahedron bounded by coordinate planes and a given plane is a fundamental problem in multivariable calculus and analytical geometry. This problem not only reinforces concepts related to three-dimensional coordinate systems but also provides insight into how planes intersect with axes to form specific geometric shapes. In this comprehensive guide, we will explore the steps involved in calculating the volume of such a tetrahedron, analyze the properties of the bounding plane, and walk through a detailed example to solidify your understanding.

Introduction to the Geometric Setup

Before diving into calculations, it's essential to understand the geometric configuration described:

- The tetrahedron is bounded by the coordinate planes (the xy-plane, yz-plane, and xz-plane).
- It is also bounded by a plane given by the equation $x + 2y + 56z = 45$.

The goal is to find the volume of this tetrahedron formed within the three-dimensional space.

Understanding the Coordinate Planes and the Bounding Plane

The Coordinate Planes

In three-dimensional space, the coordinate planes are:

- The xy-plane: $z=0$
- The xz-plane: $y=0$
- The yz-plane: $x=0$

These planes intersect at the origin $(0,0,0)$ and divide space into octants. The tetrahedron in question lies in the positive octant where $x, y, z \geq 0$.

The Plane $x + 2y + 56z = 45$

This plane intersects the coordinate axes at points where two variables are zero, and the third is found by solving the equation:

- x-intercept: set $y=0, z=0$:
$$
x + 2(0) + 56(0) = 45 \Rightarrow x=45
$$
So, the x-intercept is at $(45, 0, 0)$.

- y-intercept: set $x=0, z=0$:
$$
0 + 2y + 0 = 45 \Rightarrow y=\frac{45}{2} = 22.5
$$
So, the y-intercept is at $(0, 22.5, 0)$.

- z-intercept: set $x=0, y=0$:
$$
0 + 0 + 56z=45 \Rightarrow z=\frac{45}{56} \approx 0.8036
$$
So, the z-intercept is at $(0, 0, \frac{45}{56})$.

These intercepts define the vertices of the tetrahedron along the axes.

Vertices of the Tetrahedron

The tetrahedron is bounded by:

- The origin $O=(0,0,0)$
- The x-intercept point $A=(45, 0, 0)$
- The y-intercept point $B=(0, 22.5, 0)$
- The z-intercept point $C=\left(0, 0, \frac{45}{56}\right)$

The volume of the tetrahedron can be computed using the position vectors of these vertices.

Calculating the Volume of the Tetrahedron

Method Overview

The most straightforward approach involves using the scalar triple product of vectors emanating from a common vertex, typically the origin. The formula for the volume V of a tetrahedron with vertices at the origin O and points A, B, C is:

$$
V = \frac{1}{6} \left| \vec{OA} \cdot (\vec{OB} \times \vec{OC}) \right|
$$

Where:

- $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$ are position vectors of points A, B, C respectively.

Step-by-step Calculation

1. Express the vectors:

$$
\vec{OA} = (45, 0, 0)
$$
$$
\vec{OB} = (0, 22.5, 0)
$$
$$
\vec{OC} = \left(0, 0, \frac{45}{56} \right)
$$

2. Compute the cross product $\vec{OB} \times \vec{OC}$:

$$
\vec{OB} \times \vec{OC} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 22.5 & 0 \\
0 & 0 & \frac{45}{56}
\end{vmatrix}
$$

Calculating determinant:

- $\mathbf{i}$ component:
$$
(22.5)(\frac{45}{56}) - (0)(0) = 22.5 \times \frac{45}{56}
$$
- $\mathbf{j}$ component:

\[
- \left(0 \times \frac{45}{56} - 0 \times 0 \right) = 0
\]
- \(\mathbf{k} \) component:
\[
0 \times 0 - 22.5 \times 0 = 0
\]

Thus,

\[
\vec{OB} \times \vec{OC} = \left(22.5 \times \frac{45}{56}, 0, 0 \right)
\]

Compute the scalar:

\[
22.5 \times \frac{45}{56} = \frac{22.5 \times 45}{56}
\]

Calculate numerator:

\[
22.5 \times 45 = (22.5 \times 45)
\]

Break down:

\[
22.5 \times 45 = (20 + 2.5) \times 45 = 20 \times 45 + 2.5 \times 45 = 900 + 112.5 = 1012.5
\]

Therefore,

\[
\vec{OB} \times \vec{OC} = \left(\frac{1012.5}{56}, 0, 0 \right)
\]

3. Compute the dot product \(\vec{OA} \cdot (\vec{OB} \times \vec{OC}) \):

\[
(45, 0, 0) \cdot \left(\frac{1012.5}{56}, 0, 0 \right) = 45 \times \frac{1012.5}{56} + 0 + 0
\]

Calculate numerator:

\[
45 \times 1012.5 = (45 \times 1012.5)
\]

Break down:

$$
45 \times 1012.5 = (40 + 5) \times 1012.5 = 40 \times 1012.5 + 5 \times 1012.5 = 40,500 + 5,062.5 = 45,562.5
$$

Now divide by 56:

$$
\frac{45,562.5}{56}
$$

Perform division:

- Approximate:

$$
56 \times 813 = 56 \times 800 + 56 \times 13 = 44,800 + 728 = 45,528
$$

Remaining:

$$
45,562.5 - 45,528 = 34.5
$$

Since $56 \times 0.6 = 33.6$, and $56 \times 0.615 = 34.44$, close to 34.5.

Thus,

$$
\frac{45,562.5}{56} \approx 813 + 0.615 \approx 813.615
$$

4. Compute the volume:

$$
V = \frac{1}{6} \times | \text{scalar triple product} | \approx \frac{1}{6} \times 813.615 \approx 135.6025
$$

Therefore, the volume of the tetrahedron is approximately 135.6 cubic units.

Note: For exact calculations, leaving the volume in fractional form might be preferable, but the decimal approximation suffices for most practical purposes.

Alternative Methods for Volume Calculation

While the scalar triple product method is straightforward, other techniques include:

- Integration over the region bounded by the planes.
- Using the formula for tetrahedra with known vertices.

Integration Approach:

Set up a triple integral over the region in the positive octant bounded by the plane:

$$
z \leq \frac{45 - x - 2y}{56}
$$

with x and \(

Frequently Asked Questions

How do you find the volume of the tetrahedron bounded by the coordinate planes and the plane x + 2y + 56z = 45?

First, find the intercepts with the axes by setting two variables to zero: x-intercept at (45, 0, 0), y-intercept at (0, 22.5, 0), and z-intercept at (0, 0, 45/56). The volume of the tetrahedron is then (1/6) × (x-intercept) × (y-intercept) × (z-intercept).

What are the intercepts of the plane x + 2y + 56z = 45 with the coordinate axes?

The x-intercept is at (45, 0, 0), the y-intercept at (0, 22.5, 0), and the z-intercept at (0, 0, 45/56).

How is the volume of the tetrahedron calculated using the intercepts?

Using the intercepts, the volume V = (1/6) × x-intercept × y-intercept × z-intercept = (1/6) × 45 × 22.5 × (45/56).

What is the numerical value of the volume of the

tetrahedron bounded by the coordinate planes and the given plane?

Calculating: V = (1/6) × 45 × 22.5 × (45/56) ≈ (1/6) × 45 × 22.5 × 0.80357 ≈ 136.125 cubic units.

Why is the volume formula for this tetrahedron (1/6) times the product of intercepts?

Because the tetrahedron formed by the coordinate axes and a plane intersecting axes at points x, y, and z has a volume equal to (1/6) × x × y × z, representing a pyramid with these intercepts as vertices.

Additional Resources

Find The Volume Of The Tetrahedron Bounded By The Coordinate Planes And The Plane X + 2y + 56z = 45

Introduction

Understanding the volume of a tetrahedron formed by the intersection of coordinate planes and a given plane is a fundamental problem in multivariable calculus and spatial geometry. Such problems frequently appear in academic settings, particularly in calculus courses dealing with triple integrals, and serve as excellent exercises for visualizing the three-dimensional space and applying integration techniques.

The problem at hand involves a tetrahedron bounded by the coordinate planes xy-, yz-, and zx-planes, along with a specific plane described by the equation:

$$
x + 2y + 56z = 45
$$

This geometric figure is a classic example of a bounded polyhedron in the first octant (where $x, y, z \geq 0$). Our goal is to find its volume.

Geometric Interpretation

The Bounding Planes

- Coordinate Planes:
- xy-plane: $z=0$

- yz-plane: $x=0$
- zx-plane: $y=0$

- The Plane $x + 2y + 56z = 45$:
- This plane cuts through the first octant, forming a tetrahedral region with the coordinate planes.

Visualizing the Tetrahedron

Imagine the space in the first octant. The coordinate planes form a corner, and the given plane slices through this corner, creating a three-sided polyhedral region. The vertices of this tetrahedron are located at the points where this plane intersects the axes, found by setting two variables to zero and solving for the third.

Step-by-Step Approach to Finding the Volume

1. Find the intercepts of the plane with the axes

To determine the vertices of the tetrahedron, find where the plane intersects each of the axes:

- X-intercept:
- Set $y=0$, $z=0$:
$$
x + 2(0) + 56(0) = 45 \Rightarrow x=45
$$
- Vertex: $(45, 0, 0)$

- Y-intercept:
- Set $x=0$, $z=0$:
$$
0 + 2y + 0 = 45 \Rightarrow 2y=45 \Rightarrow y=22.5
$$
- Vertex: $(0, 22.5, 0)$

- Z-intercept:
- Set $x=0$, $y=0$:
$$
0 + 0 + 56z=45 \Rightarrow z= \frac{45}{56}
$$
- Vertex: $(0, 0, \frac{45}{56})$

The vertices of the tetrahedron are thus:

$$
A = (45, 0, 0), \quad B = (0, 22.5, 0), \quad C = (0, 0, \frac{45}{56})
$$

---

2. Formulating the Integral for Volume

The volume V of the tetrahedron can be calculated using a triple integral over the region R:

$$
V = \iiint_{R} dV
$$

The region R is bounded by the coordinate planes and the plane $x + 2y + 56z = 45$.

Expressing the Region in Spherical Coordinates

Alternatively, it is often easier to set up the limits directly in Cartesian coordinates, especially since the boundary is given explicitly.

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3. Establishing the Limits of Integration

To set up the integral, choose the order of integration. A common approach is:

- Integrate with respect to z first,
- then y,
- finally x.

This order aligns well with the shape of the region.

Step 1: Determine z-limits for fixed x and y:

From the plane equation:

$$
x + 2y + 56z = 45 \Rightarrow z = \frac{45 - x - 2y}{56}
$$

Since $z \geq 0$:

$$
z \in [0, \frac{45 - x - 2y}{56}]
$$

Step 2: Determine y-limits for fixed x:

Given $y \geq 0$ and the maximum y occurs when $z=0$ and x is fixed:

$$
x + 2y = 45 \Rightarrow 2y = 45 - x \Rightarrow y = \frac{45 - x}{2}
$$

Since $y \geq 0$:

$$
y \in [0, \frac{45 - x}{2}]
$$

Step 3: Determine x-limits:

From the x-intercept, x varies from 0 to 45:

$$
x \in [0, 45]
$$

---

4. Writing the Volume Integral

Putting all these together:

$$
V = \int_{x=0}^{45} \int_{y=0}^{\frac{45 - x}{2}} \int_{z=0}^{\frac{45 - x - 2y}{56}} dz \, dy \, dx
$$

---

5. Computing the Integral

Step 1: Integrate with respect to z:

$$
\int_{0}^{\frac{45 - x - 2y}{56}} dz = \frac{45 - x - 2y}{56}
$$

Step 2: Integrate with respect to y:

$$
V = \int_{0}^{45} \left[\int_{0}^{\frac{45 - x}{2}} \frac{45 - x - 2y}{56} dy \right] dx
$$

Pull out the constant $\frac{1}{56}$:

$$
V = \frac{1}{56} \int_0^{45} \left[\int_0^{\frac{45 - x}{2}} (45 - x - 2y) dy \right] dx
$$

\]

Now, compute the inner integral:

\[
\int_0^{\frac{45 - x}{2}} (45 - x - 2y) dy
\]

Let’s evaluate:

\[
\int (A - 2y) dy = A y - y^2
\]

where \(A = 45 - x \).

So,

\[
\int_0^{\frac{45 - x}{2}} (45 - x - 2y) dy = (45 - x) y - y^2
\Big|_0^{\frac{45 - x}{2}}
\]

Plugging in the limits:

\[
= (45 - x) \times \frac{45 - x}{2} - \left(\frac{45 - x}{2} \right)^2
\]

Simplify step by step:

\[
= \frac{(45 - x)^2}{2} - \frac{(45 - x)^2}{4} = \left(\frac{1}{2} -
\frac{1}{4} \right) (45 - x)^2 = \frac{1}{4} (45 - x)^2
\]

Thus, the volume integral simplifies to:

\[
V = \frac{1}{56} \int_0^{45} \frac{1}{4} (45 - x)^2 dx = \frac{1}{224}
\int_0^{45} (45 - x)^2 dx
\]

---

6. Final Integration and Calculation

Perform the last integral:

\[
\int_0^{45} (45 - x)^2 dx

\]

Let \(u = 45 - x \), then \(du = - dx \).

When \(x=0 \), \(u=45 \).

When \(x=45 \), \(u=0 \).

Changing limits accordingly:

\[
\int_{u=45}^{0} u^2 (-du) = \int_0^{45} u^2 du
\]

Compute:

\[
\int_0^{45} u^2 du = \frac{u^3}{3} \Big|_0^{45} = \frac{45^3}{3}
\]

Calculate \(45^3 \):

\[
45^3 = 45 \times 45 \times 45 = (45 \times 45) \times 45 = 2025 \times 45
\]

\[
2025 \times 45 = (2000 + 25) \times 45 = 2000 \times 45 + 25 \times 45 =
90,000 + 1,125 = 91,125
\]

Then,

\[
\frac{45^3}{3} = \frac{91,125}{3} = 30,375
\]

Putting it all together:

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