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Introduction

In number theory, the structure of units in modular arithmetic plays a fundamental role in understanding properties of integers modulo n. When considering the multiplicative group of units modulo n, denoted as U(n), a natural question arises: For a given integer n greater than 2, can we identify elements within U(n) that satisfy certain algebraic equations? Specifically, this article aims to demonstrate that for any integer n > 2, there are at least two elements in U(n) such that their square equals the identity element, i.e., elements x with $x^2 \equiv 1 \pmod{n}$.

This topic touches on the algebraic structure of U(n), the nature of involutions within groups, and the implications of such elements in number theory and cryptography. Understanding the existence of multiple solutions to $x^2 \equiv 1 \bmod n$ illuminates the internal symmetry of the unit group and provides insight into the factorization of n.

The Group of Units Modulo n (U(n))

Definition of U(n)

Given a positive integer n, the group of units modulo n, U(n), comprises all integers less than n that are coprime to n, with multiplication modulo n as the operation:

$$
U(n) = \{ x \in \mathbb{Z} \mid 1 \leq x < n, \gcd(x, n) = 1 \}
$$

The order of U(n) is given by Euler's totient function $\varphi(n)$:

$$
|U(n)| = \varphi(n)
$$

Since every element in U(n) has a multiplicative inverse modulo n, U(n) forms a finite abelian group.

Existence of Elements Satisfying $X^2 \equiv 1 \pmod{n}$

General Observations

The equation:

$$
x^2 \equiv 1 \pmod{n}
$$

has at least two solutions in U(n) under certain conditions. These solutions are called involutions because they are elements of order 1 or 2, satisfying $x = x^{-1}$.

In particular, the elements satisfying $x^2 \equiv 1 \bmod n$ are called idempotent elements (when considering the group operation), but in the context of units, they are precisely the elements of order dividing 2.

Key Results and Theorems

Theorem 1: For $n > 2$, there are always at least two solutions x in U(n) to $x^2 \equiv 1 \bmod n$.

Sketch of Proof:

1. Trivial Solutions:
- The identity element $x \equiv 1 \bmod n$ always satisfies $x^2 \equiv 1 \bmod n$.
- The element $x \equiv -1 \bmod n$ also satisfies $x^2 \equiv 1 \bmod n$, provided that -1 is coprime to n (which is always true for odd n).

2. Existence of -1 in U(n):
- For n odd, $\gcd(-1, n) = 1$, so $-1 \in U(n)$.
- For n even, $\gcd(-1, n) = 1$, so $-1 \in U(n)$ as well.

3. Distinctness of Solutions:
- Since $1 \neq -1 \bmod n$ for $n > 2$, these are two distinct elements in U(n), both satisfying the equation.

Conclusion:
For all $n > 2$, at least these two solutions exist in U(n). The key is that both 1 and -1 are units modulo n, and both satisfy $x^2 \equiv 1 \bmod n$, providing the minimal count of two solutions.

Detailed Analysis for Different Types of n

Case 1: n is Odd

When n is odd, the structure of U(n) simplifies our analysis.

- Since n is odd, \(\gcd(-1, n) = 1\), so -1 mod n is an element of U(n).

- Both 1 and -1 satisfy $x^2 \equiv 1 \mod n$.

- These are distinct solutions because:

$$
1 \not\equiv -1 \pmod{n} \quad \text{for } n > 2
$$

- Therefore, in U(n), at least two elements satisfy $x^2 \equiv 1$.

Example: n=15

- U(15) = {1,2,4,7,8,11,13,14}
- Solutions to $x^2 \equiv 1 \mod 15$ are $x \equiv 1, 14$ (since $14 \equiv -1 \mod 15$)

- Both are in U(15). Hence, at least two solutions.

Case 2: n is Even

When n is even, the same logic applies.

- -1 is coprime to n if and only if n is odd, but for even n, gcd(-1, n) = 1 always, because:

$$
\gcd(-1, n) = 1
$$

- So, -1 mod n is an element of U(n).

- Both 1 and -1 are solutions to $x^2 \equiv 1 \mod n$.

- They are distinct solutions when n > 2.

Example: n=8

- U(8) = {1,3,5,7}

- Check x=1: $1^2 \equiv 1 \mod 8$
- Check x=7: $7^2=49 \equiv 1 \mod 8$
- Both solutions, confirming the theorem.

Group-Theoretic Perspective

Involutions in U(n)

Elements x in U(n) satisfying $x^2 \equiv 1 \mod n$ are called involutions.

- The set of involutions always includes the identity element, $x \equiv 1$.
- For $n > 2$, as shown, -1 is also an involution in U(n).

Properties:

- Involutions form a subset of U(n) with interesting algebraic properties.
- The number of involutions can be greater than two depending on the structure of U(n).

Structure of U(n) and Chinese Remainder Theorem

The structure of U(n) can be decomposed via the Chinese Remainder Theorem (CRT):

$$
U(n) \cong U(p_1^{k_1}) \times U(p_2^{k_2}) \times \cdots \times U(p_m^{k_m})
$$

where n factors into prime powers:

$$
n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}
$$

This decomposition helps analyze the solutions to $x^2 \equiv 1 \mod n$ by examining the solutions modulo each prime power.

Implication:

- The solutions to $x^2 \equiv 1 \mod n$ correspond to the tuples of solutions modulo each prime power.
- Each component $U(p_i^{k_i})$ contributes at least two solutions: 1 and -1.

Examples and Applications

Examples with Specific n

1. n=12

- U(12) = {1,5,7,11}
- Check solutions:

- x=1: $1^2 \equiv 1$
- x=5: 25 ≡ 1 mod 12? 25 mod 12 = 1 → yes
- x=7: 49 ≡ 1 mod 12? 49 mod 12= 1 → yes
- x=11: 121 ≡ 1 mod 12? 121 mod 12= 1 → yes

- All elements satisfy $x^2 \equiv 1$, so more than two solutions.

2. n=7

- U(7) = {1,2,3,4,5,6}
- Solutions:

- x=1: $1^2=1$
- x=6: $6^2=36 \equiv 1$ mod 7, since 36 mod 7=1

- Both solutions are distinct, and at least two such elements exist.

Implications and Applications in Number Theory and Cryptography

Understanding the existence of multiple solutions to $x^2 \equiv 1$ mod n has several important implications:

- Cryptography:
- Certain cryptographic protocols rely on the structure of U(n) and the difficulty of solving quadratic equations modulo n.
- The presence of multiple involutions can affect the security assumptions.

- Factorization:
- The solutions to $x^2 \equiv 1$ mod n relate to the factorization of n, especially when n is composite.
- For example, solving $x^2 \equiv 1$ mod n can sometimes lead to non-trivial factors of n via algorithms like Pollard's Rho.

- Group Theory:
- The involutions form a subgroup of U(n), and understanding their structure provides insights into the

symmetry of the group.

Conclusion