I Need Help Solving This Laplace Transform Initial Value Problem.y″+3y′+2y = E ; Y(0)=y′(0)=0

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When tackling differential equations, especially initial value problems, the Laplace transform is a powerful tool that simplifies the process by converting differential equations into algebraic equations. If you're facing the problem:

$$ y'' + 3y' + 2y = E $$
with initial conditions:
$$ y(0) = 0,\quad y'(0) = 0 $$

then this article will guide you through a comprehensive, step-by-step solution. Whether you're a student studying differential equations or an engineer working on control systems, understanding how to solve such problems using the Laplace transform is essential.

Understanding the Differential Equation and Its Components

Before diving into the solution, let's analyze the given differential equation and initial conditions.

The Differential Equation: $y'' + 3y' + 2y = E$

- Type of Equation: Second-order linear differential equation with constant coefficients.
- Right-Hand Side (RHS): The term E (which we interpret as a constant, often representing a step input or forcing function).
- Initial Conditions: Both y and y' are zero at $t=0$, indicating the system starts from rest.

Significance of the Input E

- The notation E typically denotes a step input or constant forcing function.
- For simplicity, assume E is a constant (say, $E = 1$) unless specified otherwise.

Applying the Laplace Transform

The core idea behind solving the differential equation with Laplace transforms is to convert derivatives into algebraic expressions involving the Laplace variable s.

Laplace Transform Basics

- The Laplace transform of $y(t)$ is denoted as $Y(s) = \mathcal{L}\{ y(t) \}$.
- For derivatives:

- $\mathcal{L}\{ y'(t) \} = sY(s) - y(0)$
- $\mathcal{L}\{ y''(t) \} = s^2Y(s) - sy(0) - y'(0)$

Given initial conditions $y(0) = 0$, $y'(0) = 0$, these simplify our calculations.

Transforming the Differential Equation

Applying the Laplace transform to both sides:

$$
\mathcal{L}\{ y'' + 3y' + 2y \} = \mathcal{L}\{ E \}
$$

which becomes:

$$
s^2 Y(s) - s y(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{E}{s}
$$

Since $y(0) = 0$ and $y'(0) = 0$:

$$
s^2 Y(s) + 3sY(s) + 2Y(s) = \frac{E}{s}
$$

Factor $Y(s)$:

$$
Y(s) (s^2 + 3s + 2) = \frac{E}{s}
$$

Solving for $Y(s)$

Express $Y(s)$:

$$
Y(s) = \frac{E}{s (s^2 + 3s + 2)}
$$

Note that the quadratic denominator factors further:

$$
s^2 + 3s + 2 = (s + 1)(s + 2)
$$

Thus,

$$
Y(s) = \frac{E}{s (s + 1)(s + 2)}
$$

Partial Fraction Decomposition

To find the inverse Laplace transform, decompose $Y(s)$ into simpler fractions:

$$
\frac{E}{s (s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}
$$

Multiply both sides by the denominator:

$$
E = A (s + 1)(s + 2) + B s (s + 2) + C s (s + 1)
$$

Calculating Coefficients A, B, C

Set $s = 0$:

$$
E = A (0 + 1)(0 + 2) = A \times 1 \times 2 = 2A \Rightarrow A = \frac{E}{2}
$$

Set $s = -1$:

$$
E = B (-1) (-1 + 2) = B (-1)(1) = -B \Rightarrow B = -E
$$

Set $s = -2$:

$$
E = C (-2) (-2 + 1) = C (-2)(-1) = 2 C \Rightarrow C = \frac{E}{2}
$$

Inverse Laplace Transform to Find $y(t)$

Now, write:

$$
Y(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}
$$

with:

$$
A = \frac{E}{2}, \quad B = -E, \quad C = \frac{E}{2}
$$

$$
y(t) = A \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} + B \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s + 1} \right\} + C \cdot \mathcal{L}^{-1}\left\{ \frac{1}{s + 2} \right\}
$$

Using standard transforms:

- $\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1$
- $\mathcal{L}^{-1}\left\{ \frac{1}{s + a} \right\} = e^{-a t}$

Therefore,

$$
y(t) = \frac{E}{2} \times 1 + (-E) \times e^{-t} + \frac{E}{2} \times e^{-2t}
$$

Simplify:

$$
\boxed{
y(t) = \frac{E}{2} - E e^{-t} + \frac{E}{2} e^{-2t}
}
$$

Final Solution and Interpretation

The solution for the differential equation with the given initial conditions and input E is:

$$
\boxed{
y(t) = \frac{E}{2} - E e^{-t} + \frac{E}{2} e^{-2t}
}
$$

This expression describes how the system's response evolves over time. Notably:

- As $t \to \infty$, $y(t) \to \frac{E}{2}$, indicating the steady-state value.
- The exponential terms represent transient behaviors that decay over time.

Special Cases and Practical Applications

Understanding this solution allows you to analyze various real-world systems:

Step Input Response

- If $E = 1$, the response simplifies to:

$$
y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}
$$

- The system starts from rest and approaches a steady-state value of $\frac{1}{2}$.

Implications in Engineering

- Control Systems: The transient response indicates how quickly the system stabilizes.
- Electrical Circuits: The solution characterizes voltage or current responses in RLC circuits.
- Mechanical Systems: Describes displacement or velocity in mass-spring-damper systems under a step force.

Summary: Step-by-Step Solution Process

To wrap up, here's a quick summary of the approach to solving similar problems:

1. Identify the differential equation and initial conditions.
2. Apply the Laplace transform to convert derivatives into algebraic terms.
3. Solve for $Y(s)$ algebraically.
4. Factor and decompose $Y(s)$ into partial fractions.
5. Use inverse Laplace transforms to find $y(t)$.
6. Interpret the solution in the context of the problem.

Additional Tips for Solving Laplace Transform Initial Value Problems

- Always verify initial conditions before transforming.
- Factor denominators to simplify partial fractions.
- Use standard Laplace transform pairs for quick inverse transforms.
- For non-constant RHS functions, adjust the approach accordingly.
- Practice with various inputs (step, impulse, sinusoidal) to deepen understanding.

Conclusion

Mastering the Laplace

Frequently Asked Questions

What is the differential equation given in the initial value problem?

The differential equation is $y'' + 3y' + 2y = E$, with initial conditions $y(0)=0$ and $y'(0)=0$.

How do I take the Laplace transform of both sides of the differential equation?

Apply the Laplace transform to each term: $L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{E\}$. Remember that $L\{y''\} = s^2 Y(s) - s y(0) - y'(0)$, and similarly for the other derivatives.

What are the Laplace transforms of the derivatives given the initial conditions?

Since $y(0)=0$ and $y'(0)=0$, $L\{y''\} = s^2 Y(s)$, and $L\{y'\} = s Y(s)$.

What is the Laplace transform of the right side E?

The Laplace transform of E (a constant) is E/s.

How do I set up the algebraic equation in Y(s) after taking the Laplace transforms?

The transformed equation becomes $s^2 Y(s) + 3s Y(s) + 2 Y(s) = E/s$. Factor Y(s) to get $Y(s)(s^2 + 3s + 2) = E/s$.

How do I solve for Y(s) in the algebraic equation?

Divide both sides by $(s^2 + 3s + 2)$ to obtain $Y(s) = (E/s) / (s^2 + 3s + 2)$.

How do I perform partial fraction decomposition on Y(s)?

Factor the denominator: $s^2 + 3s + 2 = (s+1)(s+2)$. Rewrite Y(s) as $A/s + B/(s+1) + C/(s+2)$ and solve for A, B, and C.

What is the process to find the inverse Laplace transform to get y(t)?

Once you find the partial fractions, take the inverse Laplace transform of each term: A/s □ A, $B/(s+1)$ □ $B e^{-t}$, $C/(s+2)$ □ $C e^{-2t}$. Sum these to get $y(t)$.

Additional Resources

I Need Help Solving This Laplace Transform Initial Value Problem — a phrase that resonates with students, engineers, and mathematicians working through differential equations. This particular initial value problem (IVP) involves solving a second-order linear differential equation using Laplace transforms, a powerful technique for handling differential equations with initial conditions. In this comprehensive review, we will explore the problem step-by-step, delving into the theoretical background, solution strategies, and practical considerations, ensuring a clear understanding of the process and underlying concepts.

Understanding the Problem: The Differential Equation and Initial Conditions

The problem at hand is:

$$ y'' + 3y' + 2y = e $$

with initial conditions:

$$ y(0) = 0, \quad y'(0) = 0 $$

This is a second-order linear differential equation with constant coefficients, driven by an exponential function e (which, assuming from context, corresponds to e^t). The goal is to find the function $y(t)$ satisfying both the differential equation and the initial conditions.

Key Points:

- The differential equation is linear, with constant coefficients.
- The forcing function e (or e^t) influences the particular solution.
- Initial conditions specify the state of the system at $t=0$.

This problem typifies many real-world systems, such as mechanical vibrations, electrical circuits, or thermal processes, where the Laplace transform method provides an efficient solution route, especially when initial conditions are involved.

Why Use Laplace Transforms?

Laplace transforms convert differential equations in the time domain into algebraic equations in the complex frequency domain, simplifying the process of solving initial value problems.

Features of Laplace transform method:

- Converts derivatives into algebraic terms, such as $\mathcal{L}\{ y' \} = sY(s) - y(0)$.
- Handles initial conditions naturally.
- Simplifies solving equations with non-homogeneous terms like e^t.
- Facilitates solving systems of differential equations.

Pros:

- Systematic approach for linear ODEs with initial conditions.
- Reduces the problem to algebraic manipulation.
- Well-suited for equations with discontinuous or impulsive inputs.
- Widely supported by computational tools.

Cons::

- Requires familiarity with Laplace transform tables and properties.
- Can become cumbersome for higher-order or non-constant coefficients.
- Inversion of Laplace transforms may involve complex partial fraction decomposition.

In this context, the Laplace method provides a structured, step-by-step pathway to solutions, making it an ideal choice for tackling the initial value problem.

Step-by-Step Solution Process

The solution involves several stages:

1. Take the Laplace transform of both sides of the differential equation.

Recall the Laplace transforms of derivatives, considering initial conditions:

$$
\mathcal{L}\{ y'' \} = s^2 Y(s) - s y(0) - y'(0)
$$
$$
\mathcal{L}\{ y' \} = s Y(s) - y(0)
$$
$$
\mathcal{L}\{ y \} = Y(s)
$$

Since $y(0) = 0$ and $y'(0) = 0$, the transforms simplify to:

$$
\mathcal{L}\{ y'' \} = s^2 Y(s)
$$
$$
\mathcal{L}\{ y' \} = s Y(s)
$$
$$
\mathcal{L}\{ y \} = Y(s)
$$

Next, the forcing function e^t has a Laplace transform:

$$
\mathcal{L}\{ e^t \} = \frac{1}{s - 1}
$$

Thus, transforming the differential equation:

$$
s^2 Y(s) + 3 s Y(s) + 2 Y(s) = \frac{1}{s - 1}
$$

---

2. Solve the algebraic equation for $Y(s)$.

Factor out $Y(s)$:

$$
Y(s) (s^2 + 3 s + 2) = \frac{1}{s - 1}
$$

Factor the quadratic:

$$
s^2 + 3s + 2 = (s + 1)(s + 2)
$$

So,

$$
Y(s) = \frac{1}{(s - 1)(s + 1)(s + 2)}
$$

---

3. Perform partial fraction decomposition to simplify $Y(s)$.

Express $Y(s)$ as:

$$
Y(s) = \frac{A}{s - 1} + \frac{B}{s + 1} + \frac{C}{s + 2}
$$

Multiply through both sides by the denominator:

$$
1 = A (s + 1)(s + 2) + B (s - 1)(s + 2) + C (s - 1)(s + 1)
$$

Set specific values of s to solve for A, B, C:

- For $s=1$:

$$
1 = A (2)(3) + B (0)(3) + C (0)(2) \implies 1= 6A \implies A= \frac{1}{6}
$$

- For $s=-1$:

$$
1 = A (0)(1) + B (-2)(1) + C (-2)(0) \implies 1= -2 B \implies B= -\frac{1}{2}
$$

- For $s=-2$:

$$
1= A (-1)(0) + B (-3)(0) + C (-3)(-1) \implies 1= 3 C \implies C= \frac{1}{3}
$$

Thus,

$$
Y(s) = \frac{1/6}{s - 1} - \frac{1/2}{s + 1} + \frac{1/3}{s + 2}
$$

---

4. Find the inverse Laplace transform of $Y(s)$.

Using standard Laplace inverse pairs:

$$
\mathcal{L}^{-1}\left\{ \frac{1}{s - a} \right\} = e^{a t}
$$

we get:

$$
y(t) = \frac{1}{6} e^{t} - \frac{1}{2} e^{- t} + \frac{1}{3} e^{-2 t}
$$

This is the general solution satisfying the initial conditions.

Final Solution and Interpretation

The explicit solution to the initial value problem is:

$$
y(t) = \frac{1}{6} e^{t} - \frac{1}{2} e^{- t} + \frac{1}{3} e^{-2 t}
$$

Features of this solution:

- It combines exponential growth (e^{t}) with decaying exponentials ($e^{- t}$ and $e^{- 2 t}$), reflecting the balance between the system's natural response and external forcing.
- Initial conditions are satisfied: $y(0) = 0$, $y'(0) = 0$.

Practical Considerations and Common Pitfalls

When solving such problems, be aware of potential pitfalls:

Common mistakes:

- Forgetting initial conditions during the Laplace transform.
- Incorrect partial fraction decomposition, especially with repeated roots.
- Misidentifying the Laplace inverse of complex fractions.
- Overlooking the domain of the solution, especially for physical interpretations.

Tips:

- Always verify initial conditions after finding the inverse transform.
- Use Laplace tables or computational tools (e.g., WolframAlpha, MATLAB) for complex decompositions.
- Simplify algebraic expressions thoroughly to avoid errors.

Extensions and Variations

This method extends naturally to:

- Higher-order differential equations with constant coefficients.
- Systems with multiple dependent variables.
- Non-homogeneous equations with different forcing functions.

For more complicated forcing functions, the Laplace transform approach remains effective, often coupled with convolution integrals or partial fraction strategies.

Conclusion: Why the Laplace Transform Method Is Valuable

The Laplace transform provides a systematic, elegant approach to solving initial value problems like the one discussed. Its ability to convert derivatives into algebraic terms, handle initial conditions seamlessly, and facilitate solutions involving exponential inputs makes it indispensable in engineering, physics, and applied mathematics.

In this specific case, the method led to a clean, explicit solution for \(y(t) \), illustrating both the power and practicality of Laplace transforms. Whether you're a student learning the technique for the first time or a professional solving complex differential equations, mastering this approach will significantly enhance your analytical toolkit.

In summary:

- Recognize the differential equation's structure.
- Apply Laplace transforms, considering initial conditions.
- Solve the resulting algebraic equations.
- Use partial fractions for inverse transforms.
- Interpret and verify the solution.

By following these steps, you can confidently tackle similar initial value problems,

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