Let {B(t), T 0} Be A Standard Brownian Motion And X(t) = −3t+2B(t). Find The E [(X (2) + X(4))].

Introduction

Let {B(t), T 0} Be A Standard Brownian Motion And X(t) = −3t+2B(t). Find The E [(X (2) + X(4))].

This problem involves understanding the properties of standard Brownian motion (also known as Wiener process) and applying these properties to determine the expected value of a linear combination of the process at specific time points. In particular, the goal is to compute the expectation $E[X(2) + X(4)]$, given the stochastic process $X(t) = -3t + 2B(t)$.

To thoroughly analyze this problem, we will first review the fundamental concepts related to Brownian motion, the linearity of expectation, and the properties of the process $X(t)$. We will then proceed to derive the expectation step-by-step, illustrating the reasoning behind each step.

Understanding Standard Brownian Motion

Definition and Properties

Standard Brownian motion $B(t)$, for $t \geq 0$, is a continuous-time stochastic process characterized by the following properties:

- **Initial condition:** $B(0) = 0$ almost surely.
- **Independent increments:** For $0 \leq s < t$, the increment $B(t) - B(s)$ is independent of the process history up to time s.
- **Stationary increments:** The distribution of $B(t + s) - B(s)$ depends only on t, not on s.
- **Normality of increments:** $B(t) - B(s) \sim N(0, t - s)$, implying that the increments are normally distributed with mean zero and variance $t - s$.
- **Continuity:** The paths of $B(t)$ are almost surely continuous functions of t.

These properties make Brownian motion a fundamental building block in stochastic calculus, finance, physics, and many other fields.

Expected Value and Variance

Since $B(t) \sim N(0, t)$, the expectation and variance are:

$$
E[B(t)] = 0, \quad \text{Var}[B(t)] = t
$$

This linearity and known distribution simplify calculations involving expectations of functions of $B(t)$.

Analyzing the Process $X(t) = -3t + 2B(t)$

Decomposition of $X(t)$

The process $X(t)$ is a combination of a deterministic linear trend $-3t$ and a stochastic component $2B(t)$. Key points include:

- The deterministic part, $-3t$, is straightforward to analyze, with expectation $E[-3t] = -3t$.

- The stochastic part, $2B(t)$, involves scaled Brownian motion, which inherits properties from $B(t)$.

Because expectation is linear, the expectation of $X(t)$ can be expressed as:

$$
E[X(t)] = E[-3t + 2B(t)] = -3t + 2E[B(t)] = -3t + 0 = -3t
$$

This simple expression results from the linearity of expectation and the fact that $E[B(t)] = 0$.

Variance of $X(t)$

While the problem asks only for the expectation, understanding the variance provides insight into the behavior of $X(t)$:

$$
\text{Var}[X(t)] = \text{Var}[2B(t)] = 4 \text{Var}[B(t)] = 4t
$$

The deterministic component contributes no variance, as it is a fixed number.

Calculating $E[X(2) + X(4)]$

Applying Linearity of Expectation

The expectation of the sum simplifies due to the linearity of expectation:

$$
E[X(2) + X(4)] = E[X(2)] + E[X(4)]
$$

Using the earlier result for $E[X(t)]$:

$$
E[X(t)] = -3t
$$

we obtain:

$$
E[X(2)] = -3 \times 2 = -6
$$
$$
E[X(4)] = -3 \times 4 = -12
$$

Hence,

$$
E[X(2) + X(4)] = -6 + (-12) = -18
$$

This straightforward summation leverages the linearity property and the expectation of $X(t)$.

Summary of the Calculation

- Recognized the linearity of expectation to split the problem into two parts.
- Calculated the expectation of $X(t)$ at specific points using the known properties of Brownian motion.
- Summed the two expectations to find the final result.

Additional Insights and Extensions

While the problem focuses on the expectation, exploring related properties can deepen understanding.

Variance of $X(2) + X(4)$

Since $X(t)$ includes a stochastic component, the variance of the sum is also of interest:

$$
\text{Var}[X(2) + X(4)] = \text{Var}[X(2)] + \text{Var}[X(4)] + 2 \,
$$

$$
\text{Cov}[X(2), X(4)]
$$

- $\text{Var}[X(t)] = 4t$, as established earlier.
- The covariance between $X(2)$ and $X(4)$ involves the covariance of $B(2)$ and $B(4)$, which is:

$$
\text{Cov}[B(2), B(4)] = \min(2, 4) = 2
$$

since Brownian motion has covariance $\text{Cov}[B(s), B(t)] = \min(s, t)$.

Thus,

$$
\text{Cov}[X(2), X(4)] = 4 \times \text{Cov}[B(2), B(4)] = 4 \times 2 = 8
$$

Now, the variance of the sum:

$$
\text{Var}[X(2) + X(4)] = 4 \times 2 + 4 \times 4 + 2 \times 8 = 8 + 16 + 16 = 40
$$

While not required for the expectation, this illustrates the stochastic dependence structure.

Conclusion

The problem of calculating $E[X(2) + X(4)]$ demonstrates key principles in stochastic processes, especially the properties of Brownian motion and linearity of expectation. The deterministic component $-3t$ simplifies the calculation, as its expectation is straightforward. The stochastic part, scaled by a constant, has an expectation of zero, which makes the overall expectation of $X(t)$ a linear function of time.

The final answer to the problem is:

$$
\boxed{
E[X(2) + X(4)] = -18
}
$$

This result confirms the power of understanding the fundamental properties of Brownian motion and their application in expectations of more complex processes.

By mastering these concepts, one can analyze a wide range of stochastic models in finance, physics, and engineering, further illustrating the importance of Brownian motion as a foundational tool in probability theory.

Frequently Asked Questions

What is the expected value of the process X(t) = −3t + 2B(t) at a specific time t?

The expected value of X(t) is E[X(t)] = −3t + 2 E[B(t)] = −3t + 0 = −3t, since E[B(t)] = 0.

How do we compute E[X(2) + X(4)] for the given process?

Since expectation is linear, E[X(2) + X(4)] = E[X(2)] + E[X(4)]. Using the expectation of X(t), this equals (−3 2) + (−3 4) = −6 + (−12) = −18.

What properties of Brownian motion are used to find the expectation of X(t)?

The key property used is that E[B(t)] = 0 for standard Brownian motion, which simplifies the expectation calculation of X(t).

Does the variance of X(t) affect the calculation of E[X(2) + X(4)]?

No, variance does not affect the expected value; expectation depends only on the mean, which is determined by the deterministic part of X(t).

What is the final expected value of (X(2) + X(4)) for the given process?

The expected value is −18, calculated as E[X(2)] + E[X(4)] = −6 + (−12) = −18.

Additional Resources

Stochastic Processes and Expectation Computations: Analyzing the Linear Combination of Brownian Motion

When delving into the realm of stochastic processes, particularly Brownian motion, understanding how linear transformations and combinations influence expectations is fundamental. The problem at hand involves a standard Brownian motion \(\{ B(t), t \geq 0 \} \), a process renowned for its continuous paths, stationary independent increments, and Gaussian distribution properties. Our goal is to analyze the process \(X(t) = -3t + 2B(t) \) and compute the expectation \(E[X(2) + X(4)] \).

This exploration offers a comprehensive view into key concepts such as properties of Brownian motion, linearity of expectation, and the calculation of moments, providing valuable insights for students, researchers, and practitioners working with stochastic models. Let's unfold the problem systematically.

Understanding the Components: Brownian Motion and the Process $X(t)$

What is Standard Brownian Motion?

Standard Brownian motion, often denoted as $B(t)$, is a foundational stochastic process characterized by the following properties:

- Initial Condition: $B(0) = 0$ almost surely.
- Independent Increments: For any $0 \leq s < t$, the increment $B(t) - B(s)$ is independent of the process history before s.
- Stationary Increments: The distribution of $B(t) - B(s)$ depends only on $t - s$, not on the specific times s and t.
- Gaussian Increments: For each $s < t$, the increment $B(t) - B(s) \sim N(0, t - s)$, a normal distribution with mean zero and variance $t - s$.
- Continuity of Paths: The sample paths $t \mapsto B(t)$ are continuous with probability 1.

These properties make Brownian motion a versatile model for phenomena in physics, finance, and other fields where randomness evolves continuously over time.

The Process $X(t) = -3t + 2B(t)$

The process $X(t)$ is a linear transformation of Brownian motion with an added deterministic drift:

$$
X(t) = -3t + 2B(t).
$$

This form combines a deterministic component $-3t$ with a scaled stochastic component $2B(t)$. Such processes are common in modeling scenarios where a baseline trend is affected by random fluctuations, such as stock prices with a drift and volatility component.

Key Properties of $X(t)$

Before computing expectations, it's essential to understand the statistical properties of $X(t)$:

Linearity of Expectation

The expectation operator is linear, which means:

$$
E[aX + bY] = aE[X] + bE[Y],
$$

for any random variables X, Y and constants a, b.

Distribution of $X(t)$

Given that $B(t) \sim N(0, t)$, the scaled process $2B(t) \sim N(0, 4t)$. Therefore:

$$
X(t) = -3t + 2B(t) \sim N(-3t, 4t),
$$

since adding a deterministic shift $-3t$ shifts the mean, and the variance remains $4t$.

Calculating $E[X(2) + X(4)]$: Step-by-Step Approach

The task is to find:

$$
E[X(2) + X(4)].
$$

Given the linearity of expectation:

$$
E[X(2) + X(4)] = E[X(2)] + E[X(4)].
$$

Calculating each separately:

Step 1: Find $E[X(2)]$

Since $X(t) \sim N(-3t, 4t)$, the expectation is simply the mean:

$$
E[X(2)] = -3 \times 2 = -6.
$$

Step 2: Find $E[X(4)]$

Similarly,

$$
E[X(4)] = -3 \times 4 = -12.
$$

Step 3: Summing the Results

Adding these gives:

$$
E[X(2) + X(4)] = -6 + (-12) = -18.
$$

Deeper Insights and Generalization

While the calculation here is straightforward due to the Gaussian properties and linearity of expectation, this process exemplifies fundamental principles in stochastic calculus:

- Linearity: Expectation of sums decomposes into sums of expectations.
- Gaussian Distributions: The mean and variance are directly obtainable from the process definitions.
- Properties of Brownian Motion: The independence and stationarity of increments enable understanding of the joint distributions of $B(t)$ at different times.

For more complex functions of $B(t)$, such as quadratic forms or non-linear transformations, advanced techniques like Itô calculus or moment-generating functions are often required.

Additional Considerations and Applications

Variance of $X(t)$

Knowing the variance helps in understanding the process's dispersion:

$$
\text{Var}[X(t)] = 4t,
$$

which grows linearly with t, reflecting increasing uncertainty over time.

Covariance Structure

The covariance between $X(s)$ and $X(t)$ for $s < t$ is:

$$
\text{Cov}[X(s), X(t)] = \text{Cov}[2B(s), 2B(t)] = 4 \text{Cov}[B(s), B(t)].
$$

Since $B(s)$ and $B(t)$ are jointly Gaussian with:

$$
\text{Cov}[B(s), B(t)] = s,
$$

we have:

$$
\text{Cov}[X(s), X(t)] = 4s.
$$

This structure indicates that $X(t)$ is a Gaussian process with a specific covariance pattern, which can be useful for modeling and simulation.

Practical Applications

Understanding expectations of linear combinations of Brownian motion processes is vital in various fields:

- Financial Mathematics: Modeling asset prices, where the expected return involves linear drifts and stochastic volatility.
- Physics: Describing particle diffusion with external forces or drift.
- Engineering: Signal processing with noise components modeled as Brownian motion.

Conclusion

The expectation $E[X(2) + X(4)]$, derived straightforwardly from the properties of Gaussian distributions and the linearity of expectation, yields a simple but profound insight: the expected value of this combination is -18.

This analysis exemplifies how stochastic calculus tools—especially properties of Brownian motion—enable precise analytical computations. Recognizing the Gaussian nature of $X(t)$ and exploiting linearity simplifies what could be a complex problem into an elegant solution. Whether in theoretical research or practical modeling, these principles underpin much of the modern understanding of stochastic processes.

In essence, the problem underscores that even in the realm of randomness, mathematical structure provides clarity, guiding us through the probabilistic landscape with precision and confidence.

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