Let V Be A Vector Space, And Let V,WV Be A Basis For V. Prove That V+W,V+2w Is A Basis For V.

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Introduction

In linear algebra, the concept of bases plays a fundamental role in understanding the structure of vector spaces. A basis allows us to express any vector uniquely as a linear combination of basis vectors, providing a coordinate system for the space. This article explores a specific problem: Given a vector space V and two bases V and W for V, we aim to prove that the subspaces $V + W$ and $V + 2w$ (for some $w \in W$) form a basis for V. The problem involves understanding how the sum of subspaces interacts with basis elements and how scalar multiples influence the basis properties.

Preliminaries and Definitions

Vector Spaces and Bases

A vector space V over a field $\mathbb{F}$ (such as $\mathbb{R}$ or $\mathbb{C}$) is a set equipped with two operations:

- Vector addition: $V \times V \to V$
- Scalar multiplication: $\mathbb{F} \times V \to V$

A basis for V is a set of vectors $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ such that:

1. Linear independence: No vector in $\mathcal{B}$ can be written as a linear combination of the others.
2. Spanning: Every vector $v \in V$ can be expressed as a linear combination of vectors in $\mathcal{B}$.

Subspaces and Sum of Subspaces

Given subspaces $U, W \subseteq V$, their sum is defined as:

$$
U + W = \{ u + w \mid u \in U, w \in W \}
$$

This sum is itself a subspace of V. The sum of two subspaces may or may not be direct; if $U \cap W = \{0\}$, then $U + W$ is a direct sum, denoted $U \oplus W$.

The Problem Statement

Suppose:

- V is a vector space over $\mathbb{F}$.
- V and W are bases for V. (Note: The problem as stated seems to have a typographical error; typically, one basis is for V, and the other is a subset or subspace. For clarity, assume V is a basis for V, and W is a subspace spanned by a basis $\{w_1, w_2, ..., w_m\} \subseteq V$.)

In particular, for the purpose of this proof, assume:

- $V = \operatorname{span}\{v_1, v_2, ..., v_n\}$
- $W = \operatorname{span}\{w_1, w_2, ..., w_m\}$

and that $W \subseteq V$. The goal is to prove that the subspaces $V + W$ and $V + 2w$ (for some fixed $w \in W$) form a basis for V.

Understanding the Spaces $V + W$ and $V + 2w$

The Sum $V + W$

Since both V and W are subspaces of V, their sum $V + W$ is just V itself, because:

$$
V + W \subseteq V \quad \text{and} \quad V \subseteq V + W
$$

Thus,

$$
V + W = V
$$

This simplifies our problem significantly because $V + W$ is just V.

The Space $V + 2w$

The notation $V + 2w$ suggests the span of V and the vector $2w$. Since $2w \in W \subseteq V$, the space $V + 2w$ is also just V, because adding a vector already in V does not enlarge the space:

$$
V + 2w = V
$$

Clarifying the Context

Given this, the problem reduces to showing that the set $\{V + W, V + 2w\}$ forms a basis for V. Since both are essentially the same subspace V, this is trivial; the set contains the same subspace twice.

However, in typical linear algebra problems, a more meaningful interpretation involves considering the sets or collections of vectors, perhaps:

- The basis $V = \{v_1, ..., v_n\}$
- A subspace $W = \operatorname{span}\{w_1, ..., w_m\} \subseteq V$
- The vectors w being one of the basis vectors w_j
- The claim that the sets $V + W$ and $V + 2w$ (viewed as vector sets or subspaces) form bases under certain conditions

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Reformulating the Problem

Assuming the problem intends to examine the basis properties of the sets:

- $\mathcal{B}_1 = \{v_1, ..., v_n, w_1, ..., w_m\}$ (or a similar set)

- $\mathcal{B}_2 = \{v_1, \ldots, v_n, 2w\}$

and to prove that these form bases of V, the core idea involves demonstrating:

1. Linear independence of the combined or modified sets
2. Spanning of V

Step-by-Step Proof Strategy

Step 1: Confirm the sets are subsets of V

Since V and W are bases (or subspaces) of V, the vectors involved are in V. Scalar multiples like $2w$ are also in V.

Step 2: Show the sets are linearly independent

- For the set $\{v_1, \ldots, v_n, w\}$:

Since v_i form a basis and w is in W, which is independent of the v_i, provided w is outside the span of v_i, the set is linearly independent.

- For the set $\{v_1, \ldots, v_n, 2w\}$:

Because scalar multiples preserve independence, the set remains linearly independent if w is outside the span of v_i.

Step 3: Show the sets span V

- Since the original basis $v_1, \ldots, v_n$ spans V, adding w or $2w$ (provided $w \notin \operatorname{span}\{v_i\}$) will not reduce the span; it either enlarges it or maintains it.

- If $W \subseteq V$ is such that w is outside the span of v_i, then:

$$
\operatorname{span}\{v_1, \ldots, v_n, w\} = V
$$

Similarly, replacing w by $2w$ does not affect the span.

Formal Proof

Theorem

Let V be a vector space over $\mathbb{F}$, with basis $\{v_1, \ldots, v_n\}$. Let $W \subseteq V$ be a subspace with basis $\{w_1, \ldots, w_m\}$, such that $w \in W$ is outside the span of $v_1, \ldots, v_n$. Then, the sets:

1. $\mathcal{B}_1 = \{v_1, \ldots, v_n, w\}$
2. $\mathcal{B}_2 = \{v_1, \ldots, v_n, 2w\}$

are bases for V.

Proof

Step 1: Show linear independence

- $\mathcal{B}_1$:

Suppose $a_1 v_1 + \dots + a_n v_n + b w = 0$.

Since v_i form a basis, the only solution to:

$$
a_1 v_1 + \dots + a_n v_n = -b w
$$

depends on whether w is in the span of v_i. By assumption, w is outside this span, so:

- If $b \neq 0$, then w is in the span of v_i, which contradicts the assumption.
- Thus, $b = 0$, and consequently, $a_1 = \dots = a_n =$

Frequently Asked Questions

What is the significance of the sets V+W and V+2w in the context of vector spaces?

V+W and V+2w are sums of subspaces or sets within a vector space V, where V is a vector space, W is a subspace, and 2w indicates scalar multiplication of a vector w by 2. The question explores whether these sets can form bases for V under certain conditions.

Given that V and W are bases for V, how do their properties influence the basis of V+W?

Since V and W are bases for V, they are both linearly independent and span V. Their sum V+W then potentially spans V if W is a subspace or related subset, and their independence properties help determine if their union can form a basis.

What does it mean for V+W and V+2w to be bases for V?

It means that both sets are linearly independent and span V, thereby satisfying the criteria for a basis. The problem requires proving these properties hold for V+W and V+2w.

How do we prove that V+W is a basis for V?

To prove V+W is a basis, show that V+W spans V and that V+W is linearly independent. Since V and W are bases, their sum will span V if W is contained within V, and independence can be verified accordingly.

What is the role of scalar multiplication in establishing that V+2w is a basis for V?

Scalar multiplication by 2 of the vector w affects independence and span. To show V+2w is a basis, demonstrate that adding 2w maintains linear independence and that the set still spans V.

Is W necessarily a subspace of V in this context, and how does that impact the proof?

Typically, for V+W to be a meaningful sum and basis, W is considered a subspace or subset of V. This assumption simplifies showing that their sum spans V and is linearly independent.

How do linear independence and spanning relate in proving that a set is a basis?

A set is a basis if it is linearly independent and spans the vector space. The proof involves verifying both these properties for the sets V+W and V+2w.

Can V+W and V+2w be different types of bases for V? How are they similar or different?

Both are sets intended to be bases for V, but V+W involves the sum of two sets (or subspaces), while V+2w involves scalar multiplication of a vector. The proof shows both can serve as bases under the right conditions.

What are the key steps in proving that V+W and V+2w are bases for V?

The key steps are: (1) Show that each set spans V, (2) Show that each set is linearly independent, and (3) Confirm that these conditions hold for both V+W and V+2w, thus establishing they are bases.

Why is the linear independence of V+2w important in this proof?

Linear independence of V+2w ensures that adding the vector 2w does not introduce dependencies, which is essential for the set to qualify as a basis for V.

Additional Resources

Let V Be A Vector Space, And Let V,WV Be A Basis For V. Prove That V+W,V+2w Is A Basis For V.

The exploration of vector spaces and their bases is fundamental in linear algebra, providing the building blocks for understanding more complex structures in mathematics and applied sciences. In particular, the behavior of subspaces generated by specific vectors and how their combinations form bases is an area of active investigation. The statement under scrutiny—"Let V be a vector space, and let V, W be a basis for V. Prove that V+W, V+2w is a basis for V"—offers a compelling insight into the interplay between subspaces and scalar multiples of vectors within a basis. This article endeavors to dissect this statement in detail, providing a comprehensive, analytical, and instructive discourse.

Understanding the Foundations: Vector Spaces and Bases

Before engaging with the core proof, it is essential to establish a clear understanding of the fundamental concepts involved: vector spaces, bases, and subspace addition.

What Is a Vector Space?

A vector space V over a field F (commonly the real or complex numbers) is a set equipped with two operations: vector addition and scalar multiplication. These operations must satisfy certain axioms, including associativity, commutativity of addition, distributivity, existence of additive identity and inverses, and compatibility of scalar multiplication.

In essence, vector spaces are mathematical structures that generalize the familiar Euclidean space $\mathbb{R}^n$, capturing the essence of linear combinations, span, and linear independence.

What Is a Basis?

A basis of a vector space V is a set of vectors that are linearly independent and span the entire space V. Specifically:

- Linear independence means no vector in the set can be expressed as a linear combination of the others.
- Spanning indicates that every vector in V can be written as a linear combination of the basis vectors.

This concept ensures that a basis provides a minimal "coordinate system" for the vector space, allowing every vector to be uniquely represented.

Subspaces and Their Sum

Given two subspaces U and W of V, their sum $U + W$ is defined as:

$$ U + W = \{ u + w \mid u \in U, w \in W \} $$

The sum $U + W$ is itself a subspace of V. When U and W are subspaces generated by particular vectors, understanding the structure of their sum is crucial for basis analysis.

Deciphering the Problem: Statement and Intuition

The statement under review involves the following:

- V is a vector space.
- V, W form a basis for V (which appears to be a typographical error, as V cannot be a basis for itself; the intended meaning is likely that V is a vector space, and W is a subspace of V, with some basis involved).
- The goal is to prove that the sets $V + W$ and $V + 2w$ form a basis for V.

Given the structure, it appears there might be a slight ambiguity or a typographical inconsistency, but the core idea is to examine how certain subspaces constructed from V and W, and scaled vectors like $2w$, can form a basis for V.

Assumption: The problem likely intends for W to be a subspace of V, and w to be a vector in W, with the goal to analyze the bases formed by $V + W$ and $V + 2w$.

Intuitive understanding:
- Since V is a basis for itself, it is the entire space, so any sum involving V remains V.
- Exploring the sum $V + W$ involves understanding how the subspace W interacts with V.
- The scalar multiple $2w$ suggests a focus on how scalar multiplication affects the basis structure, especially when considering linear independence.

Clarifying the Precise Statement and Goal

Given the possible ambiguities, let's restate a likely intended problem:

Restated Problem:
Let V be a vector space, W be a subspace of V with basis W = {w}. Suppose V is a basis for V (which trivially holds as V itself). Show that the subspaces V + W and V + 2w are bases for V.

Key points to clarify:

- If V is a basis for itself, then V + W = V + W (which is V, assuming W ⊆ V).
- The focus then shifts to whether the set V + 2w (which is just V, since adding a vector from W scaled by 2 to V won't extend the space if W ⊆ V).

Alternatively, perhaps the problem is to analyze the sets:

- V + W
- V + 2w

and determine whether these sets constitute bases for V.

Given this, the core idea is:

- If W is a subspace of V generated by a vector w, then V + W = V, since V already contains W.
- Scaling w by 2 (to form 2w) doesn't change the subspace generated by W, but perhaps the problem is exploring how scalar multiples influence basis properties.

In conclusion, the problem's core is to analyze whether the sets V + W and V + 2w are bases for V, given the conditions.

Detailed Proof and Analytical Approach

Approach Outline:
1. Establish the context and assumptions.
2. Analyze the properties of the sets V + W and V + 2w.
3. Prove that these sets are bases for V, or clarify the intended statement.

Step 1: Clarifying the Assumptions

Suppose:

- V is a vector space over a field F.
- W is a subspace of V, generated by a vector w, so W = span{w}.
- Since V is a basis for V, it is the entire space.

The key is to examine the sets:

- V + W: the sum of V with W. Since W is a subspace of V, then V + W = V, because adding a subspace W contained in V to V doesn't enlarge V.

- V + 2w: this set involves adding the vector 2w to V. Since V already contains W, and thus w, scalar multiples like 2w are also in W, and W is contained in V. Therefore, V + 2w = V.

Step 2: Understanding the Sets V + W and V + 2w

- V + W = V:
Because W ⍰ V, the sum of V and W is just V itself.
This is a fundamental property: for any subspace W of V, V + W = V.

- V + 2w:
Since 2w ⍰ W ⍰ V, adding 2w to V yields no new vectors: V + 2w = V.

Thus, both sets are essentially equal to V, which is a basis for itself.

Step 3: Establishing the Basis Property

Given the above, the sets V + W and V + 2w are both equal to V, which is trivially a basis for V. Therefore, the statement reduces to verifying the basis property of V itself, which is vacuously true.

However, the original problem seems to aim at understanding how the addition of specific vectors or subspaces preserves or creates a basis structure. To explore this more meaningfully, consider an alternative, more general scenario:

Generalized Scenario: V as a vector space, W as a subspace generated by w, and scalar multiples of w

Suppose:

- V is a vector space over a field F.
- W = span{w} is a one-dimensional subspace of V.
- The set $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

Question:
Does the set V + W or V + 2w form a basis for V? Since V is already a basis, the addition of W or 2w won't change the span.

Key Insights and Theoretical Implications

1. The Sum of Subspaces and Bases

- For subspaces U and W of V, the sum U + W is a subspace containing both U and W.
- If U is a basis for V, then adding any subspace W contained in U doesn't change the span: U + W = U.

2. Scalar Multiplication and Basis Preservation

- Multiplying a basis vector w by a non-zero scalar (like 2) doesn't affect the span: span{w} = span{2w}.
- Therefore, adding 2w to a basis (or considering the subspace generated by 2w) maintains the same span.

3. Equivalence of Sets in Terms of Basis Formation

- The sets V + W and V + 2w both span V, and their linear independence depends on how they are constructed.

Conclusion and Final Remarks