Minimize The Function F(x,y,z)=x^2+y^2+z^2 Subject To The Constraints X+2y+3z=6 And X+3y+9z = 9

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When dealing with optimization problems in mathematics, especially those involving multiple variables and constraints, the method of Lagrange multipliers often provides a powerful tool for finding the extrema (minimum or maximum values). In this article, we will explore how to minimize the function $F(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints:

- $X + 2y + 3z = 6$
- $X + 3y + 9z = 9$

This problem is a classic example of constrained optimization, where the goal is to find the point(s) on the intersection of the given constraints that minimize the value of the function F.

Understanding the Problem

Objective Function

The function to be minimized is:

$$F(x, y, z) = x^2 + y^2 + z^2$$

This function represents the squared Euclidean distance from the point (x, y, z) to the origin. Minimizing this function essentially finds the point on the feasible region defined by the constraints that is closest to the origin.

Constraints

The constraints are linear equations:

1. $x + 2y + 3z = 6$
2. $x + 3y + 9z = 9$

These define a feasible region (likely a line or a plane intersection) in three-dimensional space.

Approach to Solving the Problem

To find the minimum of $F(x, y, z)$ subject to the constraints, we employ the method of Lagrange multipliers, which involves setting up a Lagrangian function incorporating the constraints.

Why Use Lagrange Multipliers?

Lagrange multipliers allow us to find stationary points of a function subject to equality constraints. The method involves introducing multipliers (λ and μ) for each constraint and solving the resulting system of equations.

Constructing the Lagrangian

Define the Lagrangian:

$$
\mathcal{L}(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda (x + 2y + 3z - 6) + \mu (x + 3y + 9z - 9)
$$

where λ and μ are the Lagrange multipliers associated with the constraints.

Step-by-Step Solution

1. Write down the Lagrangian

$$
\mathcal{L} = x^2 + y^2 + z^2 + \lambda (x + 2y + 3z - 6) + \mu (x + 3y + 9z - 9)
$$

2. Compute the partial derivatives

Set the derivatives with respect to x, y, z to zero:

$$
\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda + \mu = 0
$$

\[

\frac{\partial \mathcal{L}}{\partial y} = 2y + 2\lambda + 3\mu = 0
\]
\[
\frac{\partial \mathcal{L}}{\partial z} = 2z + 3\lambda + 9\mu = 0
\]

3. Solve the system of equations

We now have three equations:

\[
(1) \quad 2x + \lambda + \mu = 0
\]
\[
(2) \quad 2y + 2\lambda + 3\mu = 0
\]
\[
(3) \quad 2z + 3\lambda + 9\mu = 0
\]

Along with the original constraints:

\[
(4) \quad x + 2y + 3z = 6
\]
\[
(5) \quad x + 3y + 9z = 9
\]

Express x, y, z in terms of λ and μ:

From (1):

\[
x = -\frac{\lambda + \mu}{2}
\]

From (2):

\[
2y + 2\lambda + 3\mu = 0 \implies y = -\lambda - \frac{3}{2} \mu
\]

From (3):

\[
2z + 3\lambda + 9\mu = 0 \implies z = -\frac{3}{2} \lambda - \frac{9}{2} \mu
\]

Substitute into the constraints:

Equation (4):

$$
x + 2y + 3z = 6
$$

becomes:

$$
-\frac{\lambda + \mu}{2} + 2 \left(-\lambda - \frac{3}{2} \mu \right) + 3 \left(-\frac{3}{2} \lambda - \frac{9}{2} \mu \right) = 6
$$

Simplify each term:

$$
-\frac{\lambda + \mu}{2} - 2\lambda - 3\mu - \frac{9}{2}\lambda - \frac{27}{2}\mu = 6
$$

Combine like terms:

$$
\left(-\frac{\lambda}{2} - 2\lambda - \frac{9}{2}\lambda \right) + \left(-\frac{\mu}{2} - 3\mu - \frac{27}{2}\mu \right) = 6
$$

Express all with common denominators:

$$
\left(-\frac{\lambda}{2} - \frac{4\lambda}{2} - \frac{9\lambda}{2} \right) + \left(-\frac{\mu}{2} - \frac{6\mu}{2} - \frac{27\mu}{2} \right) = 6
$$

Simplify:

$$
\left(-\frac{1+4+9}{2} \lambda \right) + \left(-\frac{1+6+27}{2} \mu \right) = 6
$$

$$
-\frac{14}{2} \lambda - \frac{34}{2} \mu = 6
$$

$$
-7 \lambda - 17 \mu = 6
$$

Similarly, for equation (5):

\[
x + 3y + 9z = 9
\]

Substitute the expressions:

\[
-\frac{\lambda + \mu}{2} + 3 \left(- \lambda - \frac{3}{2} \mu \right) + 9 \left(- \frac{3}{2} \lambda - \frac{9}{2} \mu \right) = 9
\]

Simplify each term:

\[
-\frac{\lambda + \mu}{2} - 3 \lambda - \frac{9}{2} \mu - \frac{27}{2} \lambda - \frac{81}{2} \mu = 9
\]

Combine like terms:

\[
\left(-\frac{\lambda}{2} - 3 \lambda - \frac{27}{2} \lambda \right) + \left(-\frac{\mu}{2} - \frac{9}{2} \mu - \frac{81}{2} \mu \right) = 9
\]

Express with common denominators:

\[
\left(-\frac{\lambda}{2} - \frac{6\lambda}{2} - \frac{27\lambda}{2} \right) + \left(-\frac{\mu}{2} - \frac{9\mu}{2} - \frac{81 \mu}{2} \right) = 9
\]

Simplify:

\[
-\frac{1+6+27}{2} \lambda - \frac{1 + 9 + 81}{2} \mu = 9
\]

\[
-\frac{34}{2} \lambda - \frac{91}{2} \mu = 9
\]

\[
-17 \lambda - 45.5 \mu = 9
\]

Now, we have a system of two equations:

\[

$$
-7 \lambda - 17 \mu = 6
$$

$$
-17 \lambda - 45.5 \mu = 9
$$

Multiply the first equation by 17 to align coefficients:

$$
(-7 \lambda)(17) - (17 \mu)(17) = 6 \times 17
$$

$$
-119 \lambda - 289 \mu = 102
$$

Now, write the second equation:

$$
-17 \lambda - 45.5 \mu = 9
$$

Multiply this equation by 7:

$$
-119 \lambda - 318.5 \mu = 63
$$

Subtract the first from the

Frequently Asked Questions

What is the main goal when minimizing the function $F(x,y,z) = x^2 + y^2 + z^2$ subject to the given constraints?

The goal is to find the point (x, y, z) that satisfies the constraints $X + 2y + 3z = 6$ and $X + 3y + 9z = 9$ while minimizing the value of $F(x, y, z) = x^2 + y^2 + z^2$.

How can Lagrange multipliers be used to solve this constrained minimization problem?

Lagrange multipliers involve setting up a function combining the original function and constraints with multipliers, then solving the resulting system of equations to find critical points that may be minima.

What are the steps to solve for the minimum of F(x,y,z) under the given constraints?

First, formulate the Lagrangian function with multipliers, then compute partial derivatives, set them to zero, and solve the resulting equations along with the constraints to find candidate solutions.

What are the constraint equations in this problem?

The constraints are $X + 2y + 3z = 6$ and $X + 3y + 9z = 9$.

Is this problem solvable using substitution or elimination methods?

Yes, because the constraints are linear equations, substitution or elimination can be used to express variables and reduce the problem to a single variable minimizing the function.

What is the geometric interpretation of minimizing F(x,y,z) under these constraints?

It corresponds to finding the point on the intersection of the two planes closest to the origin, since $F(x,y,z) = x^2 + y^2 + z^2$ measures the distance squared from the origin.

Can the constraints be combined into a single equation to simplify the problem?

Yes, by subtracting one constraint from the other, we can find relationships between variables, reducing the problem's complexity.

What is the significance of the solution in practical applications?

The solution identifies the point closest to the origin that satisfies the constraints, which can be useful in optimization problems like least-distance solutions in engineering or data fitting.

What is the expected outcome after solving the problem?

The outcome is the specific coordinates (x, y, z) that minimize the function F while satisfying the constraints, along with the minimum value of F at that point.

How do the constraints influence the minimum value of the function F?

The constraints restrict the feasible region, so the minimum of F is found at the closest point to the origin within the intersection of the given planes, shaping the solution space.

Additional Resources

Minimize The Function F(x,y,z)=x^2+y^2+z^2 Subject To The Constraints X+2y+3z=6 And X+3y+9z=9

In the realm of mathematical optimization, problems often revolve around finding the minimum or maximum values of a function under specific constraints. One such problem is to minimize the function $F(x, y, z) = x^2 + y^2 + z^2$, which essentially measures the squared distance of a point (x, y, z) from the origin, subject to two linear constraints:

$$
\begin{cases}
x + 2y + 3z = 6 \\
x + 3y + 9z = 9
\end{cases}
$$

This problem is not just an abstract mathematical exercise; it has practical applications in fields like engineering, economics, and data science, where optimizing a quantity within certain limits is crucial. The goal here is to identify the point (x, y, z) that is as close as possible to the origin while satisfying both constraints.

In this article, we'll explore the process of solving this constrained optimization problem through systematic methods, such as Lagrange multipliers, and interpret the results in a clear and accessible manner.

Understanding the Problem: The Geometric Perspective

Before diving into the mathematics, it's beneficial to visualize the problem's geometric context. The function $F(x, y, z) = x^2 + y^2 + z^2$ represents the squared Euclidean distance from any point (x, y, z) to the origin $(0, 0, 0)$. Minimizing this function under given constraints is equivalent to finding the point on the intersection of the constraints' surfaces that is closest to the origin.

The Constraints as Planes

The two constraints:

- $x + 2y + 3z = 6$
- $x + 3y + 9z = 9$

are equations of planes in three-dimensional space. The intersection of these planes is generally a line, and the minimal distance point from the origin lying on this line is what we're seeking.

Key Geometric Insight

- The problem reduces to projecting the origin onto the line of intersection of the two planes.
- The shortest distance from the origin to the intersection line corresponds to the minimal value of F.

Understanding this geometric picture guides the algebraic solution and provides intuition about the problem's structure.

Formulating the Solution: The Method of Lagrange Multipliers

To systematically solve the problem, the method of Lagrange multipliers is a powerful tool. It allows us to find extrema of a function subject to equality constraints by introducing auxiliary variables.

Step 1: Set Up the Lagrangian

Define the Lagrangian function:

$$
\mathcal{L}(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda (x + 2y + 3z - 6) + \mu (x + 3y + 9z - 9)
$$

where:

- λ and μ are Lagrange multipliers corresponding to each constraint.

Step 2: Compute the Partial Derivatives

Take partial derivatives of $\mathcal{L}$ with respect to x, y, z, and set them equal to zero:

$$
\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda + \mu = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial y} = 2y + 2\lambda + 3\mu = 0
$$
\[

$$
\frac{\partial \mathcal{L}}{\partial z} = 2z + 3\lambda + 9\mu = 0
$$

Step 3: Formulate the System of Equations

The derivatives give us a system:

$$
\begin{cases}
2x + \lambda + \mu = 0 \\
2y + 2\lambda + 3\mu = 0 \\
2z + 3\lambda + 9\mu = 0
\end{cases}
$$

Alongside the original constraints:

$$
\begin{cases}
x + 2y + 3z = 6 \\
x + 3y + 9z = 9
\end{cases}
$$

Our task is to solve this system for x, y, z, λ, μ.

Solving the System: Step-by-Step Approach

Step 4: Express Variables in Terms of Multipliers

From the first equation:

$$
2x + \lambda + \mu = 0 \Rightarrow x = -\frac{\lambda + \mu}{2}
$$

Similarly, from the second:

$$
2y + 2\lambda + 3\mu = 0 \Rightarrow y = -\lambda - \frac{3\mu}{2}
$$

From the third:

$$
2z + 3\lambda + 9\mu = 0 \Rightarrow z = -\frac{3\lambda + 9\mu}{2}
$$

Step 5: Substitute into the Constraints

Plug the expressions for x, y, z into the constraints:

1. $x + 2y + 3z = 6$

$$
-\frac{\lambda + \mu}{2} + 2\left(-\lambda - \frac{3\mu}{2}\right) + 3\left(-\frac{3\lambda + 9\mu}{2}\right) = 6
$$

Simplify step-by-step:

$$
-\frac{\lambda + \mu}{2} - 2\lambda - 3\mu - \frac{9\lambda + 27\mu}{2} = 6
$$

Bring all to a common denominator (2):

$$
-\frac{\lambda + \mu}{2} - \frac{4\lambda}{2} - \frac{6\mu}{2} - \frac{9\lambda + 27\mu}{2} = 6
$$

Combine numerators:

$$
\frac{ -(\lambda + \mu) - 4\lambda - 6\mu - 9\lambda - 27\mu }{2} = 6
$$

Numerator:

$$
- \lambda - \mu - 4\lambda - 6\mu - 9\lambda - 27\mu = (- \lambda - 4\lambda - 9\lambda) + (- \mu - 6\mu - 27\mu) = (-14\lambda) + (-34\mu)
$$

So:

$$
\frac{-14\lambda - 34\mu}{2} = 6 \Rightarrow -14\lambda - 34\mu = 12
$$

Divide through by 2:

$$
-7\lambda - 17\mu = 6
$$

2. $x + 3y + 9z = 9$

Similarly, substitute:

$$
-\frac{\lambda + \mu}{2} + 3\left(-\lambda - \frac{3\mu}{2}\right) + 9\left(-\frac{3\lambda + 9\mu}{2}\right) = 9
$$

Simplify:

$$
-\frac{\lambda + \mu}{2} - 3\lambda - \frac{9\mu}{2} - \frac{27\lambda + 81\mu}{2} = 9
$$

Express all over denominator 2:

$$
\frac{ -(\lambda + \mu) - 6\lambda - 9\mu - 27\lambda - 81\mu }{2} = 9
$$

Combine numerator:

$$
- \lambda - \mu - 6\lambda - 9\mu - 27\lambda - 81\mu = (- \lambda - 6\lambda - 27\lambda) + (- \mu - 9\mu - 81\mu) = (-34 \lambda) + (-91 \mu)
$$

Thus:

$$
\frac{-34 \lambda - 91 \mu}{2} = 9 \Rightarrow -34 \lambda - 91 \mu = 18
$$

Solving for λ and μ

Now, we have a system of two linear equations:

$$
\begin{cases}
-7\lambda - 17\mu = 6 \\
-34 \lambda - 91 \mu = 18
\end{cases}
$$

Notice that the second equation is a multiple of the first:

$$
-34 \lambda - 91 \mu = 2 \times (-17 \lambda - 17 \mu) = 2 \times 6 = 12
$$

But according to our earlier calculation, it's equal to 18, which indicates

```
an inconsistency unless:

\[
-34 \lambda - 91 \mu = 18
\]
and
\[
-17 \lambda - 17 \mu = 6
\]

Multiplying
```

Minimize The Function F X Y Z X 2 Y 2 Z 2 Subject To The Constraints X 2y 3z 6 And X 3y 9z 9

Minimize The Function F X Y Z X 2 Y 2 Z 2 Subject To The Constraints X 2y 3z 6 And X 3y 9z 9

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