Problem 02: Find The Exact Arc Length Of The Curve X=1/8y4+1/4y^2 Over The Interval Y=1 To Y=4.

Problem 02: Find The Exact Arc Length Of The Curve X=1/8y^4 + 1/4y^2 Over The Interval Y=1 To Y=4.

Understanding how to determine the exact length of a curve is a fundamental concept in calculus, particularly in the study of analytic geometry and integral calculus. The problem at hand requires calculating the precise arc length of a given curve $x = \frac{1}{8} y^4 + \frac{1}{4} y^2$ over the interval $y = 1$ to $y = 4$. This article aims to guide you through the process with clarity, detailed explanations, and step-by-step calculations, making it accessible to students and enthusiasts alike.

Introduction to Arc Length Calculation

Before we delve into the specifics of the problem, it’s essential to understand the general methodology for calculating the arc length of a curve. When a curve is expressed as a function $x = f(y)$, the arc length L between two points $y = a$ and $y = b$ is given by the integral:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dx}{dy} \right)^2} \, dy
$$

This formula is derived from the Pythagorean theorem, considering the infinitesimal elements of the curve as segments connecting two close points. The integrand involves the derivative of x with respect to y, which measures the slope of the curve at each point.

Step-by-Step Solution to the Problem

Let's now apply this principle to our specific problem.

1. Identify the function and interval

Given:

$$
x = \frac{1}{8} y^4 + \frac{1}{4} y^2
$$

Interval:

$$
y \in [1, 4]
$$

\]

Our goal:
Calculate the exact arc length \(L\) of this curve over the specified interval.

2. Compute the derivative \(\frac{dx}{dy} \)

Differentiate \(x\) with respect to \(y\):

\[
\frac{dx}{dy} = \frac{d}{dy} \left(\frac{1}{8} y^4 + \frac{1}{4} y^2 \right)
\]

Applying power rule:

\[
\frac{dx}{dy} = \frac{1}{8} \times 4 y^3 + \frac{1}{4} \times 2 y
\]

Simplify:

\[
\frac{dx}{dy} = \frac{4}{8} y^3 + \frac{2}{4} y = \frac{1}{2} y^3 + \frac{1}{2} y
\]

Factor out \(\frac{1}{2} y \):

\[
\frac{dx}{dy} = \frac{1}{2} y (y^2 + 1)
\]

3. Compute \(\left(\frac{dx}{dy} \right)^2 \)

Square the derivative:

\[
\left(\frac{dx}{dy} \right)^2 = \left(\frac{1}{2} y (y^2 + 1) \right)^2
\]

\[
= \left(\frac{1}{2} \right)^2 y^2 (y^2 + 1)^2
\]

\[
= \frac{1}{4} y^2 (y^2 + 1)^2
\]

4. Set up the integral for arc length

The formula for the arc length:

$$
L = \int_{1}^{4} \sqrt{ 1 + \left(\frac{dx}{dy} \right)^2 } \, dy
$$

Substitute the value:

$$
L = \int_{1}^{4} \sqrt{ 1 + \frac{1}{4} y^2 (y^2 + 1)^2 } \, dy
$$

5. Simplify the integrand

Express inside the square root:

$$
\sqrt{ 1 + \frac{1}{4} y^2 (y^2 + 1)^2 }
$$

To combine terms, write 1 as a fraction with denominator 4:

$$
= \sqrt{ \frac{4}{4} + \frac{1}{4} y^2 (y^2 + 1)^2 }
$$

$$
= \sqrt{ \frac{ 4 + y^2 (y^2 + 1)^2 }{4} }
$$

Pull out the square root:

$$
= \frac{1}{2} \sqrt{ 4 + y^2 (y^2 + 1)^2 }
$$

Now, focus on simplifying the numerator expression:

$$
4 + y^2 (y^2 + 1)^2
$$

Expand $(y^2 + 1)^2$:

$$
(y^2 + 1)^2 = y^4 + 2 y^2 + 1
$$

\]

Multiply by \(y^2 \):

\[
y^2 (y^4 + 2 y^2 + 1) = y^6 + 2 y^4 + y^2
\]

Therefore, the expression inside the square root becomes:

\[
4 + y^6 + 2 y^4 + y^2
\]

Rearranged:

\[
y^6 + 2 y^4 + y^2 + 4
\]

Notice that this polynomial resembles a perfect square. Let's check if it factors as such.

6. Recognize the perfect square structure

Attempt to write:

\[
(y^3 + a y^2 + b y + c)^2
\]

But to be more straightforward, note that:

\[
(y^3 + y)^2 = y^6 + 2 y^4 + y^2
\]

Compare this to our expression:

\[
y^6 + 2 y^4 + y^2 + 4
\]

So,

\[
(y^3 + y)^2 = y^6 + 2 y^4 + y^2
\]

Adding 4 to this:

$$
(y^3 + y)^2 + 4
$$

Thus,

$$
4 + y^6 + 2 y^4 + y^2 = (y^3 + y)^2 + 2^2
$$

Recognize this as a sum of squares:

$$
A^2 + B^2
$$

which can be written as:

$$
(y^3 + y)^2 + 2^2
$$

The expression under the square root simplifies to:

$$
\sqrt{ (y^3 + y)^2 + 2^2 }
$$

Therefore, the integrand becomes:

$$
\frac{1}{2} \sqrt{ (y^3 + y)^2 + 4 }
$$

7. Final form of the integral

The arc length expression is:

$$
L = \int_{1}^{4} \frac{1}{2} \sqrt{ (y^3 + y)^2 + 4 } \, dy
$$

This integral, while appearing complex, can be approached with substitution or recognizing the structure related to inverse hyperbolic functions or trigonometric substitutions.

Advanced Approach: Substitution for the Integral

Given the form:

$$
L = \frac{1}{2} \int_{1}^{4} \sqrt{ (y^3 + y)^2 + 4 } \, dy
$$

we can set:

$$
u = y^3 + y
$$

Then:

$$
du/dy = 3 y^2 + 1
$$

which implies:

$$
du = (3 y^2 + 1) dy
$$

But our integrand involves only u, not dy directly. To proceed, express dy:

$$
dy = \frac{du}{3 y^2 + 1}
$$

The limits change accordingly:

- When $y = 1$:

$$
u = 1^3 + 1 = 2
$$

- When $y = 4$:

$$
u = 4^3 + 4 = 64 + 4 = 68
$$

Now, rewriting the integral:

$$
L = \frac{1}{2} \int_{u=2}^{68} \sqrt{ u^2 + 4 } \times \frac{dy}{du} du
$$

\]

But \(dy\) is expressed as:

\[
dy = \frac{du}{3 y^2 + 1}
\]

We need to express \(y^2 \) in terms of \(u \). From the substitution:

\[
u = y^3 + y
\]

which is a cubic in \(y \). Solving explicitly

Frequently Asked Questions

What is the main objective in Problem 02 regarding the curve $x = (1/8)y^4 + (1/4)y^2$?

The goal is to find the exact length of the arc of the given curve between $y = 1$ and $y = 4$.

Which formula is used to compute the arc length of a curve y = y in terms of x?

The arc length L is given by the integral L = ∫ from y=1 to y=4 of $\sqrt{1 + (dx/dy)^2}$ dy.

How do you find the derivative dx/dy for the given curve?

Differentiate $x = (1/8)y^4 + (1/4)y^2$ with respect to y to obtain $dx/dy = (1/2)y^3 + (1/2)y$.

What is the expression inside the square root in the arc length integral?

It is $1 + [(dx/dy)^2]$, which simplifies to $1 + [(1/2)y^3 + (1/2)y]^2$.

How can the integral for the arc length be simplified before evaluation?

Expand $(dx/dy)^2$ to get $(1/4)(y^3 + y)^2$, then add 1 to it, resulting in $\sqrt{[1 + (1/4)(y^3 + y)^2]}$.

Is the integral for the arc length straightforward to evaluate

analytically, or does it require substitution?

The integral involves a square root of a polynomial, which may require substitution or recognizing it as a standard form to evaluate exactly.

What is the significance of finding the exact arc length in this problem?

Determining the exact arc length provides precise geometric information about the curve between y=1 and y=4, which is essential in various applications such as engineering and physics.

Additional Resources

Problem 02: Find The Exact Arc Length Of The Curve X=1/8 y^4 + 1/4 y^2 Over The Interval y=1 To y=4

Introduction

Calculus, a foundational branch of mathematics, offers powerful tools for understanding the behavior of curves and shapes. One particular problem that often arises is determining the arc length of a given curve between two points. This task not only tests one's grasp of integral calculus but also involves meticulous algebraic manipulation and application of the arc length formula.

In this detailed examination, we focus on the problem: Find the exact arc length of the curve $x = \frac{1}{8} y^4 + \frac{1}{4} y^2$ over the interval $y = 1$ to $y = 4$. This problem is representative of many similar tasks in calculus where the goal is to transition from a parametric or explicit function to a precise numerical or algebraic expression for the length of a curve.

Understanding the Problem

Before delving into calculations, it is essential to understand the key components:

- Curve Equation: $x = \frac{1}{8} y^4 + \frac{1}{4} y^2$. This expresses x explicitly as a function of y.

- Interval: From $y=1$ to $y=4$.

- Objective: Find the exact length of the curve segment between these two points.

The problem's core challenge lies in correctly setting up and evaluating the integral that computes the arc length.

The Arc Length Formula for Curves

The general formula for the arc length L of a curve $x = f(y)$ between $y=a$ and $y=b$ is:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dx}{dy} \right)^2} \, dy
$$

This formula stems from the Pythagorean theorem extended infinitesimally along the curve, summing the tiny segments' lengths.

Step 1: Computing the Derivative $\frac{dx}{dy}$

Given:

$$
x = \frac{1}{8} y^4 + \frac{1}{4} y^2
$$

Differentiate with respect to y:

$$
\frac{dx}{dy} = \frac{1}{8} \times 4 y^3 + \frac{1}{4} \times 2 y = \frac{1}{2} y^3 + \frac{1}{2} y
$$

Simplify:

$$
\frac{dx}{dy} = \frac{1}{2} y^3 + \frac{1}{2} y = \frac{1}{2} (y^3 + y)
$$

Step 2: Setup of the Arc Length Integral

Substitute $\frac{dx}{dy}$ into the arc length formula:

$$
L = \int_{1}^{4} \sqrt{1 + \left(\frac{1}{2} (y^3 + y) \right)^2} \, dy
$$

Simplify the expression inside the square root:

$$
L = \int_{1}^{4} \sqrt{1 + \frac{1}{4} (y^3 + y)^2} \, dy
$$

Expressed explicitly:

$$
L = \int_{1}^{4} \sqrt{1 + \frac{1}{4} (y^3 + y)^2} \, dy
$$

Step 3: Simplify the Expression Under the Square Root

Calculate $(y^3 + y)^2$:

$$
(y^3 + y)^2 = y^6 + 2 y^4 + y^2
$$

Therefore:

$$
L = \int_{1}^{4} \sqrt{1 + \frac{1}{4} (y^6 + 2 y^4 + y^2)} \, dy
$$

Distribute $\frac{1}{4}$:

$$
L = \int_{1}^{4} \sqrt{1 + \frac{1}{4} y^6 + \frac{1}{2} y^4 + \frac{1}{4} y^2} \, dy
$$

Express as a single radical:

$$
L = \int_{1}^{4} \sqrt{ \left(1 + \frac{1}{4} y^6 + \frac{1}{2} y^4 + \frac{1}{4} y^2 \right) } \, dy
$$

Step 4: Recognizing the Perfect Square Structure

The challenge now is to recognize whether the expression inside the root can be factored or expressed as a perfect square. To do this, compare the expression to a potential square of a binomial.

Suppose:

$$
\left(a y^3 + b y \right)^2 = a^2 y^6 + 2 a b y^4 + b^2 y^2
$$

Matching terms:

$$
a^2 y^6 \leftrightarrow \frac{1}{4} y^6 \Rightarrow a^2 = \frac{1}{4} \Rightarrow a = \pm \frac{1}{2}
$$

Similarly, for the y^4 term:

\[
2 a b y^4 \leftrightarrow \frac{1}{2} y^4
\[
\Rightarrow 2 a b = \frac{1}{2} \Rightarrow b = \frac{1/2}{2 a} = \frac{1/2}{2 \times (1/2)} = \frac{1/2}{1} = \frac{1}{2}
\]

Now, check the \(y^2 \) term:

\[
b^2 y^2 \leftrightarrow \frac{1}{4} y^2 \Rightarrow b^2 = \frac{1}{4} \Rightarrow b = \pm \frac{1}{2}
\]

Since earlier \(b = \frac{1}{2} \), this is consistent.

Thus, the binomial:

\[
a y^3 + b y = \frac{1}{2} y^3 + \frac{1}{2} y
\]

Squaring:

\[
\left(\frac{1}{2} y^3 + \frac{1}{2} y \right)^2 = \frac{1}{4} y^6 + \frac{1}{2} y^4 + \frac{1}{4} y^2
\]

which matches exactly the expression under the radical.

Adding the remaining 1:

\[
1 + \frac{1}{4} y^6 + \frac{1}{2} y^4 + \frac{1}{4} y^2 = \left(\frac{1}{2} y^3 + \frac{1}{2} y \right)^2 + 1
\]

This suggests rewriting the integrand as:

\[
\sqrt{ \left(\frac{1}{2} y^3 + \frac{1}{2} y \right)^2 + 1 }
\]

Step 5: Expressing the Integral in Terms of a Sum of Squares

Now, the integral becomes:

\[

L = \int_{1}^{4} \sqrt{ \left(\frac{1}{2} y^3 + \frac{1}{2} y \right)^2 + 1 } \, dy
\]

Define:

\[
u(y) = \frac{1}{2} y^3 + \frac{1}{2} y
\]

Then:

\[
L = \int_{1}^{4} \sqrt{ u(y)^2 + 1 } \, dy
\]

Observe that:

\[
\frac{du}{dy} = \frac{1}{2} \times 3 y^2 + \frac{1}{2} = \frac{3}{2} y^2 + \frac{1}{2}
\]

Expressed explicitly:

\[
\frac{du}{dy} = \frac{3 y^2 + 1}{2}
\]

Step 6: Substitution Strategy

Since the integrand involves \(\sqrt{ u^2 + 1 } \), a hyperbolic substitution is appropriate:

\[
u = \sinh t \Rightarrow du = \cosh t \, dt
\]

Then:

\[
\sqrt{ u^2 + 1 } = \sqrt{ \sinh^2 t + 1 } = \cosh t
\]

Thus, the integral in terms of \(t \):

\[
L = \int_{t_1}^{t_2} \cosh t \times \frac{dy}{dt} \, dt
\]

But this requires expressing \(dy/dt \). Alternatively, since \(du/dy = \frac{3 y^2 + 1}{2} \), we can write:

\[
dy = \frac{2}{3 y^

Problem 02 Find The Exact Arc Length Of The Curve X 1 8y4 1 4y 2 Over The Interval Y 1 To Y 4

Problem 02 Find The Exact Arc Length Of The Curve X 1 8y4 1 4y 2 Over The Interval Y 1 To Y 4

Related Articles

- What Pressure P1 Is Needed To Produce A Flow Rate Of 0.09 Ft^3/s From The Tank. D = 0.057 Ft.
- If TWS And TWV Are Supplementary Angles, Then What Is The Measure Of TWS? A. 60 B. 70 C. 69 D. 21
- The Diameter Of A Circle Is 2 Miles. What Is The Circle's Area?d=2 Misquare MilesUse 3.14 For .

Back to Home